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Exceptional discrete mapping class group orbits in moduli spaces

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Abstract. Let M be a four-holed sphere and Γ the mapping class group of M fixing ∂M. The group Γ acts on the space $\mathscr{M}_{\mathscr{B}}(\mathrm{SU}(2))$ of SU(2)-gauge equivalence classes of flat SU(2)-connections on M with fixed holonomy on ∂M. We give examples of flat SU(2)-connections whose holonomy groups are dense in SU(2), but whose Γ-orbits are discrete in $\mathscr{M}_{\mathscr{B}}(\mathrm{SU}(2))$. This phenomenon does not occur for surfaces with genus greater than zero.

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1 Introduction

Let M be a Riemann surface of genus g with n boundary components (circles). Let

$$\{\gamma_1, \gamma_2, \dots, \gamma_n\} \subset \pi_1(M)$$

be the elements in the fundamental group corresponding to these n boundary components. Assign to each γ_i a conjugacy class $B_i \subset \mathrm{SU}(2)$ and let

$$\mathscr{B} = \{B_1, B_2, \dots, B_n\},$$

$$\mathscr{H}_{\mathscr{B}} = \{\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2)) : \rho(\gamma_i) \in B_i, 1 \leq i \leq n\}.$$

A conjugacy class in SU(2) is determined by its trace which is in $[-2, 2]$. Hence we might consider $\mathscr{B}$ as an element in $[-2, 2]^n$. The group SU(2) acts on $\mathscr{H}_{\mathscr{B}}$ by conjugation.

Definition 1.1. The *moduli space with fixed holonomy* $\mathscr{B}$ is

$$\mathscr{M}_{\mathscr{B}} = \mathscr{H}_{\mathscr{B}}/\mathrm{SU}(2).$$

Denote by $[\rho]$ the image of $\rho \in \mathscr{H}_{\mathscr{B}}$ in $\mathscr{M}_{\mathscr{B}}$. The set of smooth points of $\mathscr{M}_{\mathscr{B}}$ possesses a natural symplectic structure which gives rise to a finite measure μ on $\mathscr{M}_{\mathscr{B}}$ (see [1, 2]).

Let $\mathrm{Diff}(M, \partial M)$ be the group of diffeomorphisms of M fixing ∂M. The mapping class group Γ is $\pi_0(\mathrm{Diff}(M, \partial M))$. The group Γ acts on $\pi_1(M)$ fixing the B_i's. This action induces a Γ-action on $\mathscr{M}_{\mathscr{B}}$.

Theorem 1.2 (Goldman). *The mapping class group* Γ *acts ergodically on* $\mathscr{M}_{\mathscr{B}}$ *with respect to the measure* μ.

Since $\mathscr{M}_{\mathscr{B}}$ has a natural topology, one may also study the topological dynamics of the mapping class group action and we have [4, 5]:

Theorem 1.3. *Suppose* M *is an orientable surface with boundary and* $g > 0$. *Let* $\rho \in \mathscr{H}_{\mathscr{B}}$ *such that* $\rho(\pi_1(M))$ *is dense in* $\mathrm{SU}(2)$. *Then the* Γ*-orbit of the conjugacy class* $[\rho] \in \mathscr{M}_{\mathscr{B}}$ *is dense in* $\mathscr{M}_{\mathscr{B}}$.

In this paper we show that the analog of the above result does not hold for $g = 0$.

Theorem 1.4. *Let* M *be a four-holed sphere. Then there exists a subset* $F \subset [-2, 2]^4$ *of two real dimensions with the following property: Suppose* $\mathscr{B} \in F$. *Then there exists* $\rho \in \mathscr{H}_{\mathscr{B}}$ *with* $\rho(\pi_1(M))$ *dense in* $\mathrm{SU}(2)$, *but the* Γ*-orbit of the conjugacy class* $[\rho]$ *is discrete in* $\mathscr{M}_{\mathscr{B}}$.

Let G be a subgroup of $\mathrm{SU}(2)$. We say that a representation ρ is a G-representation if $\rho(\pi_1(M)) \subset G$. The group $\mathrm{SU}(2)$ is a double cover of $\mathrm{SO}(3)$:

$$p : \mathrm{SU}(2) \to \mathrm{SO}(3).$$

The group $\mathrm{SO}(3)$ contains $\mathrm{O}(2)$, and the symmetry groups of the regular polyhedra: $\mathscr{T}'$ (the tetrahedron), $\mathscr{C}'$ (the cube), and $\mathscr{D}'$ (the dodecahedron). Let $\mathrm{Pin}(2)$, $\mathscr{T}$, $\mathscr{C}$, and $\mathscr{D}$ denote the groups $p^{-1}(\mathrm{O}(2))$, $p^{-1}(\mathscr{T}')$, $p^{-1}(\mathscr{C}')$, and $p^{-1}(\mathscr{D}')$, respectively. The proper closed subgroups of $\mathrm{SU}(2)$ consist of $\mathscr{T}$, $\mathscr{C}$, $\mathscr{D}$, and the closed subgroups of $\mathrm{Pin}(2)$. The group $\mathrm{Pin}(2)$ has two components, and we write

$$\mathrm{Pin}(2) = \mathrm{Spin}(2) \cup \mathrm{Spin}_-(2),$$

where $\mathrm{Spin}(2)$ is the identity component of $\mathrm{Pin}(2)$.

Remark 1.5. Suppose $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$. If $\rho(\pi_1(M))$ is not contained in any of the aforementioned closed subgroups, then it is dense in $\mathrm{SU}(2)$.

We adopt the following notational conventions: For a fixed representation ρ and an element $X \in \pi_1(M)$, we write X for $\rho(X)$ when there is no ambiguity. A small letter denotes the trace of the matrix represented by the corresponding capital letter.

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2 The moduli space of the four-holed sphere

We first review some results that appear in [1, 5]. Suppose M is a three-holed sphere. Then $\pi_1(M)$ has a presentation:

$$\langle A, B, C : ABC = I \rangle,$$

where $A, B,$ and C represent the homotopy classes of the three boundaries of M.

Proposition 2.1. (1) *A representation ρ on a three-holed sphere is a* Spin(2)-*representation if and only if $a^2 + b^2 + c^2 - abc - 4 = 0$.*

(2) *A representation ρ on a three-holed sphere is a* Pin(2)-*representation and not a* Spin(2)-*representation if and only if $a^2 + b^2 + c^2 - abc - 4 \neq 0$ and at least two of the three: A, B, AB, have zero trace.*

Suppose M is a four-holed sphere. Then the fundamental group $\pi_1(M)$ admits a presentation

$$\langle A, B, C, D : ABCD = I \rangle.$$

Set $X = AB$, $Y = BC$, and $Z = CA$. Let $\kappa = (a, b, c, d) \in [-2, 2]^4$ be the holonomies on the boundary. Then the moduli space $\mathcal{M}_\kappa$ is the subspace of $[-2, 2]^3$ given by the equation [1, 3, 5]

$$\begin{aligned} &x^2 + y^2 + z^2 + xyz \\ &\quad = (ab + cd)x + (ad + bc)y + (ac + bd)z - (a^2 + b^2 + c^2 + d^2 + abcd - 4). \end{aligned}$$

Remark 2.2. [1] If two representations in $\mathcal{M}_\kappa$ share (x, y, z), then they are conjugate.

Let

$$I_{a,b} = \left[\frac{ab - \sqrt{(a^2 - 4)(b^2 - 4)}}{2}, \frac{ab + \sqrt{(a^2 - 4)(b^2 - 4)}}{2}\right].$$

If $I_{a,b} \cap I_{c,d} \neq \emptyset$, then $\mathcal{M}_\kappa$ is a (possibly degenerate) topological sphere (see Figure 1). Note that x is an endpoint of $I_{a,b}$ if and only if (x, a, b) is a reducible character.

The mapping class group Γ of the 4-holed sphere is generated by three Dehn twists τ_X, τ_Y, τ_Z [1, 5]. In local coordinates, the actions are

$$\begin{bmatrix} y \\ z \end{bmatrix} \overset{\tau_X}{\longmapsto} \begin{bmatrix} ad + bc - x(ac + bd - xy - z) - y \\ ac + bd - xy - z \end{bmatrix},$$

$$\begin{bmatrix} z \\ x \end{bmatrix} \overset{\tau_Y}{\longmapsto} \begin{bmatrix} bd + ca - y(ba + cd - yz - x) - z \\ ba + cd - yz - x \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{\tau_Z}{\longmapsto} \begin{bmatrix} cd + ab - z(cb + ad - zx - y) - x \\ cb + ad - zx - y \end{bmatrix}.$$

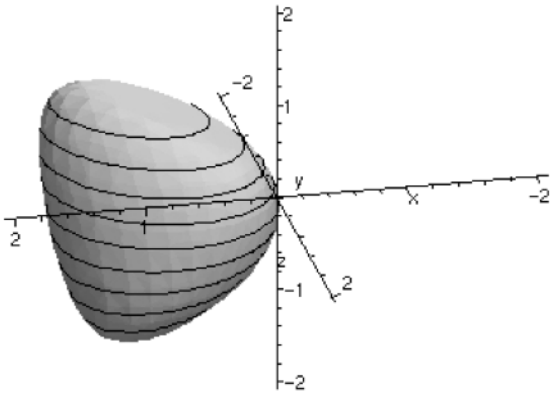

Fig. 1. The sphere $\mathscr{M}_\kappa$ for $\kappa = \left(\sqrt{2}, \sqrt{2}, \frac{1}{2}, -\frac{1}{2}\right)$.

3 The Pin(2) representations

Consider

$$e^{i\theta} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \imath = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

in Pin(2).

Proposition 3.1. *Suppose* $\rho \in \mathscr{H}_{(a,b,c,d)}$ *with* $a, b, c, d \notin \{\pm 2\}$ *and* $[\rho] = (x, y, z) \in \mathscr{M}_\kappa$. *Then the representation* ρ *is a* Spin(2)*-representation if and only if* x *is an endpoint of both* $I_{a,b}$ *and* $I_{c,d}$, y *is an endpoint of both* $I_{b,c}$ *and* $I_{a,d}$, *and* z *is an endpoint of both* $I_{a,c}$ *and* $I_{b,d}$.

Proof. First, suppose that ρ is a Spin(2)-representation. Then, up to conjugation,

$$\rho(A) = e^{i\theta_a}, \quad \rho(B) = e^{i\theta_b}, \quad \rho(C) = e^{i\theta_c}, \quad \rho(D) = e^{i\theta_d},$$

where $\theta_a + \theta_b + \theta_c + \theta_d = 0$. The endpoints of $I_{a,b}$ are given by

$$\frac{1}{2}(ab \pm \sqrt{(4 - a^2)(4 - b^2)})$$

$$= \cos(\theta_a + \theta_b) + \cos(\theta_a - \theta_b) \pm \frac{1}{2}\sqrt{(4 - 4\cos^2(\theta_a))(4 - 4\cos^2(\theta_b))}$$

$$= \cos(\theta_a + \theta_b) + \cos(\theta_a - \theta_b) \pm |\cos(\theta_a - \theta_b) - \cos(\theta_a + \theta_b)|.$$

This implies that $2\cos(\theta_a + \theta_b) = x$ is an end point of $I_{a,b}$. A similar calculation shows that x is also an end point of $I_{c,d}$. The same argument shows that y is an endpoint of $I_{b,c}$ and $I_{a,d}$, and z is an endpoint of $I_{a,c}$ and $I_{b,d}$.

To prove the converse, suppose that ρ is such that x is an endpoint of both $I_{a,b}$ and

$I_{c,d}$, y is an endpoint of both $I_{b,c}$ and $I_{a,d}$, and z is an endpoint of both $I_{a,c}$ and $I_{b,d}$. Then $2x = ab \pm \sqrt{(4-a^2)(4-b^2)}$ which implies that

$$x^2 + a^2 + b^2 - xab = 4$$

which implies that ρ is a Spin(2)-representation on the three-holed sphere (A, B, X) by Proposition 2.1. Similarly, (C, D, X), (A, C, Z), (B, D, Z), (A, D, Y), and (B, C, Y) are Spin(2)-representations. As A, B, C, and D all pairwise commute, ρ is a Spin(2)-representation on the entire four-holed sphere. □

Proposition 3.2. *Let* $\rho \in \mathscr{H}_\kappa$ *and* $[\rho] = (x, y, z) \in \mathscr{M}_\kappa$. *Suppose* ρ *is a* Pin(2)-*representation but not a* Spin(2)-*representation then one of the following hold:*

(1) $\kappa = (0,0,0,0)$

(2) $\kappa = (0,0,c,d)$, *where* $y = 0$ *and* $z = 0$,

(3) $\kappa = (0,b,0,d)$, *where* $x = 0$ *and* $y = 0$,

(4) $\kappa = (0,b,c,0)$, *where* $x = 0$ *and* $z = 0$,

(5) $\kappa = (a,0,0,d)$, *where* $x = 0$ *and* $z = 0$,

(6) $\kappa = (a,0,c,0)$, *where* $y = 0$ *and* $z = 0$,

(7) $\kappa = (a,b,0,0)$, *where* $x = 0$ *and* $y = 0$.

Moreover, a representation that satisfies any of the above conditions is a Pin(2)-*representation.*

Proof. Let ρ be a Pin(2)-representation but not a Spin(2)-representation. Then one of A, B, C, D is in $\mathrm{Spin}_-(2)$. Since $ABCD = I$, at least two of A, B, C, D is in $\mathrm{Spin}_-(2)$. Suppose without loss of generality that $A, B \in \mathrm{Spin}_-(2)$. If $C \in \mathrm{Spin}_-(2)$, then $D \in \mathrm{Spin}_-(2)$, and $\kappa = (0,0,0,0)$. If $C \in \mathrm{Spin}(2)$, then $D \in \mathrm{Spin}(2)$, which implies that $y = z = 0$.

Now let $\rho \in \mathscr{H}_{(0,0,0,0)}$. Set $A = \imath$, $B = -\imath e^{i\theta} \in \mathrm{Pin}(2)$ with x, y, z satisfying the equation $x^2 + y^2 + z^2 + xyz = 4$. Set $x = 2\cos\theta$ (in A and B above) and C equal to one of $e^{\pm i\psi}\imath$, where $z = -2\cos\psi$. A direct computation shows that $(A, B, C, (ABC)^{-1})$ is conjugate to ρ.

Now suppose that $\rho \in \mathscr{H}_{(0,0,c,d)}$ with $y = z = 0$. Thus x, c, d satisfies $x^2 = cdx - c^2 - d^2 + 4$ implying that ρ restricted to (X, C, D) is a Spin(2)-representation by Proposition 2.1. Let $A = \imath$, $B = -\imath e^{i\theta} \in \mathrm{Pin}(2)$ with $x = 2\cos\theta$ and $C = e^{i\psi}$. Then $(A, B, C, (ABC)^{-1})$ is conjugate to ρ. Similar arguments hold for the other cases. □

Propositions 3.1 and 3.2 provide a complete characterization of the Pin(2)-representation classes.

4 Examples

A direct computation shows that the traces of elements in the groups $\mathscr{C}, \mathscr{D}$ are in the set

$$S = \left\{0, \pm 1, \pm\sqrt{2}, \pm\frac{\sqrt{5}+1}{2}, \pm\frac{\sqrt{5}-1}{2}, \pm 2\right\}.$$

Let F be the set of $\kappa = (a, a, c, -c)$ (up to permutations of coordinates) $\in [-2, 2]^4$ satisfying the following conditions:

(1) $a^2 + c^2 < 4$,

(2) $a \neq 0$ and $c \neq 0$,

(3) $a \notin S$ or $c \notin S$.

Consider the space $\mathcal{M}_\kappa$ with $\kappa \in F$. The orbit

$$\mathcal{O} = \{(a^2 - 2, 0, 0), (2 - c^2, 0, 0)\} \subset \mathcal{M}_\kappa$$

is Γ-invariant. Moreover conditions (1), (2) and (3) guarantee that elements of $\mathcal{O}$ do not correspond to $\mathrm{Pin}(2), \mathcal{C}, \mathcal{D}$-representations. By Remark 1.5, the elements in the discrete orbit $\mathcal{O}$ correspond to representations with dense images in $\mathrm{SU}(2)$. This proves Theorem 1.4.

Figure 1 shows one such case with $\kappa = \left(\sqrt{2}, \sqrt{2}, \frac{1}{2}, -\frac{1}{2}\right)$. The special orbit $\mathcal{O}$ consists of the two points that are intersections of the x-axis with $\mathcal{M}_\kappa$, i.e. $\mathcal{O} = \left\{(0, 0, 0), \left(\frac{7}{4}, 0, 0\right)\right\}$. Below is a representation in the conjugacy class $(0, 0, 0) \in \mathcal{O} \subset \mathcal{M}_\kappa$:

$$A = B = \begin{bmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i & 0 \\ 0 & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \end{bmatrix} \quad \text{and} \quad C = -D = \begin{bmatrix} \frac{1}{4} + \frac{1}{4}i & \frac{\sqrt{14}}{4} \\ -\frac{\sqrt{14}}{4} & \frac{1}{4} - \frac{1}{4}i \end{bmatrix}.$$

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