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The Moduli of Flat $\mathrm{PGL}(2,\mathbb{R})$ Connections on Riemann Surfaces

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Abstract: Suppose X is a compact Riemann surface with genus $g > 1$. Each class $[\sigma] \in \mathrm{Hom}(\pi_1(X), \mathrm{PGL}(2,\mathbb{R}))/\,\mathrm{PGL}(2,\mathbb{R})$ is associated with the first and second Stiefel–Whitney classes $w_1([\sigma])$ and $w_2([\sigma])$. The set of representation classes with a fixed $w_1 \neq 0$ has two connected components. These two connected components are characterized by w_2 being 0 or 1. For each fixed $w_1 \neq 0$, we prove that the component, characterized by $w_2 = 0$, contains an open dense set diffeomorphic to the total space of a vector bundle of rank $2g-2$ over a once punctured algebraic torus of dimension $g-1$. The other component, characterized by $w_2 = 1$, contains an open dense set diffeomorphic to the total space of a vector bundle of rank $2g-2$ over an algebraic torus of dimension $g-1$.

1. Introduction

Let X be a compact Riemann surface of genus $g > 1$, and

$$\mathrm{Hom}(\pi_1(X), \mathrm{PGL}(2,\mathbb{R}))$$

the space of homomorphisms from $\pi_1(X)$ to $\mathrm{PGL}(2,\mathbb{R})$. The group $\mathrm{PGL}(2,\mathbb{R})$ has two connected components and is isomorphic to $\mathrm{SO}(2,1)$.

The space $\mathrm{Hom}(\pi_1(X), \mathrm{PSL}(2,\mathbb{R}))$ has $4g-3$ connected components and these components are distinguished by the Euler class e [6,9,10,18]. To obtain more detailed information on these representation spaces, Hitchin made use of the complex structure on X. By studying the space of rank-2 Higgs bundles over X, he showed that the $2g-2$ connected components (corresponding to non-zero Euler classes) of

$$\mathrm{Hom}(\pi_1(X), \mathrm{PSL}(2,\mathbb{R}))/\,\mathrm{PSL}(2,\mathbb{R})$$

are complex vector bundles over symmetric products of X [9].

Let $\alpha \in \mathrm{H}^1(X, \mathbb{Z}_2)$ and define

$$PW_\alpha = \{\sigma \in \mathrm{Hom}(\pi_1(X), \mathrm{PGL}(2, \mathbb{R})) : w_1(\sigma) = \alpha\}.$$

For any two non-zero classes $\alpha, \beta \in \mathrm{H}^1(X, \mathbb{Z}_2)$, PW_α is homeomorphic to PW_β [19]. Fix a non-zero class α and define PW to be PW_α. Then PW has two connected components distinguished by the two Stiefel–Whitney classes in $\mathrm{H}^2(X, \mathbb{Z}_2)$ [19].

This paper is a study of the topology of the space

$$\mathrm{Hom}(\pi_1(X), \mathrm{PGL}(2, \mathbb{R}))/\,\mathrm{PGL}(2, \mathbb{R}),$$

in particular the component $PW/\,\mathrm{PGL}(2, \mathbb{R})$. Each representation $\sigma \in PW$ may be lifted to an element $\tilde{\sigma} = \pi^*(\sigma) \in \mathrm{Hom}(\tilde{X}, \mathrm{SL}(2, \mathbb{R}))$, where

$$\pi : \tilde{X} \longrightarrow X$$

is a chosen unramified double cover of X. Let PW' be the subset of PW such that $\sigma \in PW'$ implies that σ is irreducible and $\tilde{\sigma}$ is a semi-simple but non-central representation. In particular, PW' is open and dense in PW. For a precise description of PW', see Sect. 2.

Theorem 1.1. *The space $PW'/\,\mathrm{PGL}(2, \mathbb{R})$ has two connected components $P\mathcal{Q}_0$ and $P\mathcal{Q}_1$. The component $P\mathcal{Q}_0$ is the total space of a vector bundle of rank $2g-2$ over an once punctured compact algebraic torus of dimension $g-1$. The component $P\mathcal{Q}_1$ is the total space of a vector bundle of rank $2g-2$ over an algebraic torus of dimension $g-1$.*

The precise description of these two components is given in Sect. 3 and 6.

Corollary 1.2. *The space $\mathrm{Hom}(\pi_1(X), \mathrm{PGL}(2, \mathbb{R}))$ has $2^{2g+1}+4g-5$ connected components.*

A representation is called parabolic if it is reducible but not semi-simple. The set $PW \setminus PW'$ consists of representations of three types:

1. The $\mathbb{R}^*$ representations.
2. The parabolic representations
3. The representations that lift to parabolic representations by π^*.

Together these points form a subvariety of PW.

2. The Pull-Back Representations of $\pi_1(\tilde{X})$

Let X be a compact Riemann surface of genus $g > 1$ and G an algebraic group. A representation $\sigma \in \mathrm{Hom}(\pi_1(X, x), G)$ defines a flat G-bundle P over X. Let

$$\begin{aligned} \mathrm{SL}_-(2, \mathbb{R}) &= \{g \in \mathrm{GL}(2, \mathbb{R}) : \det(g) = -1\}, \\ \mathrm{SL}_i(2, \mathbb{R}) &= \mathrm{SL}(2, \mathbb{R}) \cup i\,\mathrm{SL}(2, \mathbb{R})_-, \\ \mathrm{SL}_{\pm i}(2, \mathbb{R}) &= \mathrm{SL}_i(2, \mathbb{R}) \cup i\,\mathrm{SL}_i(2, \mathbb{R}). \end{aligned}$$

Then $\mathrm{SL}_i(2, \mathbb{R})$ is a subgroup of $\mathrm{SL}(2, \mathbb{C})$ and has two connected components. The projectivization of both $\mathrm{SL}_i(2, \mathbb{R})$ and $\mathrm{SL}_{\pm i}(2, \mathbb{R})$ is $\mathrm{PGL}(2, \mathbb{R})$.

The obstruction classes of P give rise to the obstruction class maps

$$o_n : \mathrm{Hom}(\pi_1(X,x), G) \longrightarrow \mathrm{H}^n(X, \pi_{n-1}(G)).$$

In particular, if G is $\mathrm{GL}(2,\mathbb{R})$, $\mathrm{PGL}(2,\mathbb{R})$ or $\mathrm{SL}_i(2,\mathbb{R})$, then o_1 is the first Stiefel–Whitney class w_1 [17,19]. The class o_2 is the second Stiefel–Whitney class w_2 when G is $\mathrm{PSL}(2,\mathbb{C})$ and the Euler class e when G is $\mathrm{PSL}(2,\mathbb{R})$ [6,17].

Fix a point $x \in X$. Choose a set of generators $S = \{a_i, b_i\}_{i=0}^{g-1}$ for the fundamental group $\pi_1(X,x)$ and define R to be the formal expression

$$\prod_{i=0}^{g-1} a_i b_i a_i^{-1} b_i^{-1}.$$

Then $\pi_1(X,x)$ is generated by S with the relation $R = 1$. Let Γ be the central extension of $\pi_1(X,x)$ by c with $R = c$ and $c^2 = 1$. This gives the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Gamma \longrightarrow \pi_1(X,x) \longrightarrow 1.$$

Let M be the space $\mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{C}))$ which has two connected components depending on whether c goes to I or $-I$ [2,6,9]. Denote the two components by M_0 and M_1, respectively. Note M_0 is the space $\mathrm{Hom}(\pi_1(X,x), \mathrm{SL}(2,\mathbb{C}))$. The space $N = \mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{R}))$ has $4g-3$ connected components consisting of $\{N_j\}_{j=2-2g}^{2g-2}$ [6,9] and is a subset of $\mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{C}))$. Each $\sigma \in \mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{C}))$ acts on $\mathbb{C}^2$ via the standard representation of $\mathrm{SL}(2,\mathbb{C})$.

Definition 2.1. *A representation σ is irreducible if its action on $\mathbb{C}^2$ is irreducible, and is semi-simple if it is a direct sum of irreducible representations.*

Identify $\mathrm{H}^2(X,\mathbb{Z}_2)$ with $\mathbb{Z}_2$ and $\mathrm{H}^2(X,\mathbb{Z})$ with $\mathbb{Z}$. Let $J_2(X)$ be the space of central representations:

$$J_2(X) = \mathrm{Hom}(\pi_1(X,x), \{\pm I\}) \cong \mathbb{Z}_2^{2g}.$$

Define

$$\begin{aligned} PM &= \mathrm{Hom}(\pi_1(X,x), \mathrm{PSL}(2,\mathbb{C})),\\ PM_i &= w_2^{-1}(i) \subset PM,\\ PN &= \mathrm{Hom}(\pi_1(X,x), \mathrm{PSL}(2,\mathbb{R})),\\ PN_j &= e^{-1}(j) \subset PN. \end{aligned}$$

The space $J_2(X)$ is a group and acts on the M_i's and N_i's. The quotients are precisely the PM_i's and PN_i's (the projective representations).

Remark 2.2. We shall use the superscripts s and ss to denote the irreducible and semi-simple subspaces. For example, M^s and M^{ss} denote the subspaces of irreducible and semi-simple subspaces of M, respectively.

Definition 2.3.

$$\begin{aligned} W &= \{\sigma \in \mathrm{Hom}(\Gamma, \mathrm{SL}_i(2,\mathbb{R})) :\\ &\qquad \sigma(a_0) \in i\,\mathrm{SL}_-(2,\mathbb{R}) \text{ and } \sigma(S \setminus \{a_0\}) \subset \mathrm{SL}(2,\mathbb{R})\},\\ W_0 &= \{\sigma \in W : \sigma(c) = I\},\\ W_1 &= \{\sigma \in W : \sigma(c) = -I\}. \end{aligned}$$

The subspaces W_0 and W_1 are the ones associated with the second Stiefel–Whitney class w_2 being 0 and 1, respectively. The group $J_2(X) \cap W$ acts on W. Denote by PW, PW_0, PW_1 the respective quotient spaces. The sets PW_0 and PW_1 are connected [19].

There exists a double cover $\tilde{X}$ of X with covering map [1]

$$\pi : \tilde{X} \longrightarrow X$$

such that $\pi_1(\tilde{X}, \tilde{x})$ is generated by $\tilde{S} = \{\tilde{a}_i, \tilde{b}_i\}_{i=1-g}^{g-1}$ with

$$\pi_*(\tilde{a}_0) = a_0^2,$$
$$\pi_*(\tilde{b}_0) = b_0,$$

and for $i > 0$,

$$\pi_*(\tilde{a}_i) = a_0^{-1}\pi_*(\tilde{a}_{-i})a_0 = a_i,$$
$$\pi_*(\tilde{b}_i) = a_0^{-1}\pi_*(\tilde{b}_{-i})a_0 = b_i.$$

The double cover admits a fixed point free involution τ that is π-invariant, i.e, the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tau} & \tilde{X} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ X & \xrightarrow{id} & X \end{array}$$

commutes. Composition of representations $\sigma \in PW$ with the induced map

$$\pi_* : \pi_1(\tilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x)$$

defines a map

$$\pi^* : PW \longrightarrow \mathrm{Hom}(\pi_1(\tilde{X}, \tilde{x}), \mathrm{PSL}(2, \mathbb{R})).$$

Proposition 2.4. *The image of π^* consists of representations $\tilde{\sigma}$ satisfying $e(\tilde{\sigma}) = 0$.*

Proof. Let $\tilde{P} = \pi^*(P)$. Then $\tilde{P}$ admits an involution:

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tau^*} & \tilde{P} \\ \Big\downarrow & & \Big\downarrow \\ \tilde{X} & \xrightarrow{\tau} & \tilde{X}. \end{array}$$

Since $\sigma(a_0) \in i\,\mathrm{SL}_-(2, \mathbb{R})$, the associated flat principal bundle P is not orientable. Hence τ^* must reverse orientations on $\tilde{P}$. Since τ preserves orientations on $\tilde{M}$, but τ^* reverses orientations on $\tilde{P}$,

$$e(\tilde{\sigma}) = \tau^*(e(\tilde{\sigma})) = e(\tau^*(\tilde{\sigma})) = -e(\tilde{\sigma}).$$

This implies $e(\tilde{\sigma}) = 0$. □

Corollary 2.5. *The representation $\tilde{\sigma}$ may be further lifted to a representation in* $\mathrm{Hom}(\pi_1(\tilde{X}, \tilde{x}), \mathrm{SL}(2, \mathbb{R}))$.

Proof. The obstruction to lifting is the mod-2 reduction of the Euler class $e(\tilde{\sigma})$ which is zero by Proposition 2.4. □

Let F_0 be the group $\pi_*(\pi_1(\tilde{X},\tilde{x}))$ which has index 2 in $\pi_1(X,x)$ and let $F_1 = \pi_1(X,x)\setminus F_0$. Consider the following homomorphisms of groups

$$\pi_1(\tilde{X},\tilde{x}) \xrightarrow{\pi_*} F_0 \xrightarrow{i} \pi_1(X,x).$$

These homomorphisms are injective. Therefore, a representation $\sigma \in W$ induces a representation

$$\tilde{\sigma} : \pi_1(\tilde{X},\tilde{x}) \longrightarrow \mathrm{SL}(2,\mathbb{R}).$$

This defines a map

$$\pi^* : W \longrightarrow \mathrm{Hom}(\pi_1(\tilde{X},\tilde{x}), \mathrm{SL}(2,\mathbb{R})).$$

The map π^* is equivariant with respect to the action of PGL$(2,\mathbb{R})$, thus, descends to a map

$$\pi^* : W/\,\mathrm{PGL}(2,\mathbb{R}) \longrightarrow \mathrm{Hom}(\pi_1(\tilde{X},\tilde{x}), \mathrm{SL}(2,\mathbb{R}))/\,\mathrm{PGL}(2,\mathbb{R}).$$

Definition 2.6.

$$\begin{aligned} W^s &= \{\sigma \in W : \sigma \text{ is irreducible}\},\\ W'' &= \{\sigma \in W^s : \pi^*(\sigma) \text{ is irreducible}\},\\ W' &= \{\sigma \in W^s : \pi^*(\sigma) \text{ is semi-simple, and } \sigma(a_0^2) \neq \pm I\} \cup W'',\\ PW^s &= (J_2(X)\cap W)\backslash W^s,\\ PW' &= (J_2(X)\cap W)\backslash W'. \end{aligned}$$

The subspaces associated with w_2 being 0 *and* 1 *are denoted by the subscripts* 0 *and* 1.

Proposition 2.7. *The subspace W' is open and dense in W.*

Proof. The space W^s is smooth and open and dense in W. The subvariety $W^s \setminus W'$ has real codimension at least 1 in W^s. Hence W' is open and dense in W^s and is, therefore, open and dense in W. □

Corollary 2.8. *The subspace PW' is open and dense in PW.*

Proposition 2.9. *The projection π^* is a 2-to-1 map on W' and the two points in each fibre differ by a central representation.*

Proof. Let $\sigma_1, \sigma_2 \in W'$ such that

$$\tilde{\sigma} = \pi^*(\sigma_1) = \pi^*(\sigma_2).$$

Then

$$\tilde{\sigma} \circ \pi_*(\tilde{a}_0) = \sigma_1(a_0^2) = \sigma_2(a_0^2).$$

Case (1). Suppose $\sigma_1(a_0^2) = \sigma(a_0^2) \neq \pm I$. Then there are exactly two elements $\pm A \in \mathrm{SL}(2,\mathbb{R})$ such that

$$(\pm A)^2 = \sigma_1(a_0)^2 = \sigma_1(a_0^2).$$

Hence

$$\sigma_2(a_0) = \pm A = \pm\sigma(a_0).$$

Hence the inverse image of $\tilde{\sigma}$ by π^* has two points and these two points differ by a central representation.

Case (2). Let $\sigma_1, \sigma_2 \in W''$. Then $\tilde{\sigma}$ is irreducible. Hence

$$\sigma_1 \mid_{F_0} = \sigma_2 \mid_{F_0}$$

is irreducible. Let $d \in F_0$. Then $a_0^{-1} d a_0 \in F_0$. This implies

$$\sigma_1(a_0^{-1} d a_0) = \sigma_2(a_0^{-1} d a_0).$$

Hence

$$\sigma_2(a_0)\sigma_1(a_0^{-1})\sigma_1(d) = \sigma_2(d)\sigma_2(a_0)\sigma_1(a_0^{-1}).$$

That is, $\sigma_2(a_0)\sigma_1(a_0^{-1})$ intertwines $\sigma_1|_{F_0}$. Since $\sigma_1|_{F_0}$ is irreducible, by Schur's lemma, $\sigma_2(a_0)\sigma_1(a_0^{-1})$ is in the center of $\mathrm{SL}(2,\mathbb{R})$. Thus, $\sigma_2(a_0) = \pm\sigma_1(a_0)$. □

Corollary 2.10. *The map π^* is 1-to-1 on PW'.*

Corollary 2.11. 1. *The map π^* is 2-to-1 on $W'/\,\mathrm{PGL}(2,\mathbb{R})$ and the two points in each fibre differ by a central representation.*
2. *The map π^* is 1-to-1 on $PW'/\,\mathrm{PGL}(2,\mathbb{R})$.*

3. The Prym Variety over $\tilde{X}$

Consider the given complex structure on X and denote by K its canonical bundle. The projection π induces a complex structure on $\tilde{X}$ and the free involution τ preserves this structure. Any holomorphic bundle E over X pulls back to a holomorphic bundle $\tilde{E}$ over $\tilde{X}$ such that $\tau^*(\tilde{E}) = \tilde{E}$. In particular, $\tau^*\tilde{K} = \tilde{K}$, where $\tilde{K}$ is the canonical bundle on $\tilde{X}$. Let $Div^0(X)$ denote the group of all degree zero divisors on X. The Jacobi variety $J(X)$ is the space of holomorphic line bundles over X with degree zero [1].

For any holomorphic line bundle L over X, π^*L is a holomorphic line bundle over $\tilde{X}$. Hence π induces a homomorphism

$$\pi^* : J(X) \longrightarrow J(\tilde{X}).$$

If $D \in Div^0(X)$, then $\pi^{-1}(D) \in Div^0(\tilde{X})$. The resulting homomorphism

$$\pi^* : Div^0(X) \longrightarrow Div^0(\tilde{X})$$

together with the basic epimorphism u satisfy the commutative diagram [1]:

$$\begin{array}{ccc} Div^0(X) & \xrightarrow{\pi^*} & Div^0(\tilde{X}) \\ \big\downarrow u & & \big\downarrow u \\ J(X) & \xrightarrow{\pi^*} & J(\tilde{X}) \end{array}$$

On the other hand, if $\tilde{D} \in Div^0(\tilde{X})$, then $\pi(\tilde{D}) \in Div^0(X)$. Hence π also induces a homomorphism (the norm map)

$$Nm : Div^0(\tilde{X}) \longrightarrow Div^0(X).$$

The map Nm descends to a homomorphism

$$Nm : J(\tilde{X}) \longrightarrow J(X)$$

and the diagram

$$\begin{array}{ccc} Div^0(\tilde{X}) & \xrightarrow{Nm} & Div^0(X) \\ \downarrow u & & \downarrow u \\ J(\tilde{X}) & \xrightarrow{Nm} & J(X) \end{array}$$

commutes [1].

For $\tilde{D} \in Div^0(\tilde{X})$, $\tau^{-1}(\tilde{D})$ is in $Div^0(\tilde{X})$. Hence τ induces automorphisms τ^* on the group $Div^0(\tilde{X})$ and $J(\tilde{X})$ such that the diagram

$$\begin{array}{ccc} Div^0(\tilde{X}) & \xrightarrow{\tau^*} & Div^0(\tilde{X}) \\ \downarrow u & & \downarrow u \\ J(\tilde{X}) & \xrightarrow{\tau^*} & J(\tilde{X}) \end{array}$$

commutes.

Let

$$\mathcal{P} = \{\tilde{L} \in J(\tilde{X}) : \tau^*(\tilde{L}) = -\tilde{L}\},$$
$$\mathcal{S} = \{\tilde{L} \in J(\tilde{X}) : \tau^*(\tilde{L}) = \tilde{L}\},$$

Remark 3.1. 1. The space $\mathcal{S}$ is an abelian variety of dimension g.
2. The identity component $\mathcal{P}_0$ of $\mathcal{P}$ is, by definition, the Prym Variety $Prym(\tilde{X}, \tau)$.
3. The subgroups of 2-torsions of $J(X)$ and $J(\tilde{X})$ are precisely $J_2(X)$ and $J_2(\tilde{X})$.

The group $\mathcal{P}$ contains the subgroup $J_2(\tilde{X}) \cap \mathcal{P}$ and the quotient is denoted by

$$P\mathcal{P} = (J_2(\tilde{X}) \cap \mathcal{P})\backslash\mathcal{P}.$$

Proposition 3.2. 1. *The kernel of the map $\pi^* : J(X) \longrightarrow J(\tilde{X})$ has two points, namely the trivial bundle 1 and a two torsion T_η.*
2. $|\mathcal{P} \cap J_2(\tilde{X})| = 2^{2g}$.
3. *$\mathcal{P}$ has four connected components and $P\mathcal{P}$ is connected.*
4. *$\mathcal{P}$ contains* $\mathrm{Ker}(Nm)$ *as a subgroup of index 2.*
5. *If* $\deg(\tilde{L}') = 2$ *such that $\tau^*\tilde{L}' = \tilde{L}'$, then there exists $\tilde{L}_1$ such that $\tilde{L}_1^2 = \tilde{L}'$ and $\tau^*(\tilde{L}_1) = \tilde{L}_1 \otimes \tilde{T}$, where $\tilde{T} \in \mathrm{Ker}(Nm) \setminus \mathcal{P}_0$.*

Proof. For 1, 2, 3 and 4, see [1,11]. Suppose $\deg(\tilde{L}') = 2$. Then it is immediate that there exists $\tilde{L}_1$ such that $\tilde{L}_1^2 = \tilde{L}'$. Since $\tau^*(\tilde{L}') = \tilde{L}'$,

$$(\tau^*(\tilde{L}_1))^2 = \tau^*(\tilde{L}_1^2) = \tau^*(\tilde{L}') = \tilde{L}' = \tilde{L}_1^2.$$

Hence,

$$\tau^*(\tilde{L}_1) = \tilde{L}_1 \otimes \tilde{T}$$

for some $\tilde{T} \in J_2(\tilde{X})$. In addition, since

$$\tilde{T} = \tilde{L}_1 \otimes (\tau^*(\tilde{L}_1))^{-1}$$

and $\deg(\tilde{L}_1) = 1$, $\tilde{T} \in \mathrm{Ker}(Nm) \setminus P_0$ [1]. □

4. Stable Holomorphic Pairs and the Self-Dual Equation

This section briefly reviews the rank-2 gauge theory over Riemann surfaces. The main results are due to Corlette, Hitchin and Donaldson. See [2–5,9] for details. For general smooth projective varieties, see [13–16].

4.1. The complex case. The maximum compact subgroups of $\mathrm{GL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C})$ are $\mathrm{U}(2)$ and $\mathrm{SU}(2)$ with fundamental groups isomorphic to $\mathbb{Z}_2$. Let P^c be a principal $\mathrm{GL}(2,\mathbb{C})$ bundle over a compact Riemann surface with first Chern class $c_1(P^c)$ being either 0 or 1. Let V be the associated vector bundle.

Fix a Hermitian metric h on V. This corresponds to a reduction of P^c to a $\mathrm{U}(2)$ principal bundle P over X. Choose a $\mathrm{U}(2)$ (i.e. compatible with h) connection D_0 on V such that the curvature $F(D_0)$ is central [2,9]. In addition, in the case of $c_1(P^c) = 0$, we choose h to be the constant metric 1 and $D_0 = d$. Denote by $\mathcal{G}^c$ the $\mathrm{SL}(2,\mathbb{C})$ gauge group on P^c and $\mathcal{G}$ the $\mathrm{SU}(2)$ gauge group on P. The gauge group $\mathcal{G}$ preserves h. Let

$$\mathrm{ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{su}(2), \quad \mathrm{ad}(P^c) = P^c \times_{\mathrm{Ad}} \mathfrak{sl}(2,\mathbb{C}),$$

where Ad is the adjoint representation.

The difference of any two connections on P^c or P is a 1-form. Hence, with the choice of D_0, one may identify $\Omega^1(X, \mathrm{ad}(P^c))$ and $\Omega^1(X, \mathrm{ad}(P))$ with the space of connections of the fixed determinant $\det(D_0)$ on P^c and P, respectively. An element Φ of $\Omega^{1,0}(X, \mathrm{ad}(P^c))$ is called a *Higgs field*. Given $\Phi \in \Omega^{1,0}(X, \mathrm{ad}(P^c))$ and $A \in \Omega^1(X, \mathrm{ad}(P))$, one may construct connections D_A and D:

$$D_A = D_0 + A, \quad D = D_0 + A + (\Phi + \Phi^*),$$

where Φ^* denotes the adjoint of Φ with respect to the metric h.

The $(0,1)$ part of D_0 determines a holomorphic structure $\bar{\partial}_0$ on V [7]. Again, let $A \in \Omega^1(X, \mathrm{ad}(P))$, i.e. D_A is compatible with h. Then $\bar{\partial}_A$, the $(0,1)$ part of D_A, defines a holomorphic structure on V. Similarly, given a holomorphic structure $\bar{\partial}$ on V with

$$\det(\bar{\partial}) = \det(\bar{\partial}_0),$$

there exists a unique $A \in \Omega^1(X, \mathrm{ad}(P))$ such that D_A is compatible with h and

$$\bar{\partial}_A = \bar{\partial}.$$

Hence the metric h determines a one-to-one correspondence between the space $\Omega^1(X, \mathrm{ad}(P))$ and the space of holomorphic structures on V with determinant equal to $\det(\bar{\partial}_0)$.

Higgs fields are sections of the bundle $\mathrm{End}_0\, V \otimes K$, where $\mathrm{End}_0\, V$ is the bundle of trace free complex linear transformations of V, and K is the canonical bundle on X. A holomorphic structure $\bar{\partial}$ on V induces on $\mathrm{End}_0\, V$ a holomorphic structure which, when combined with the inherent holomorphic structure on K, gives a holomorphic structure (which we shall also call $\bar{\partial}$) on $\mathrm{End}_0\, V \otimes K$. A Higgs field Φ is holomorphic if

$$\bar{\partial}\Phi = 0.$$

When Φ is holomorphic, we say $(\bar{\partial}, \Phi)$ is a *Higgs bundle*. A pair (D_A, Φ) is *holomorphic* if

$$\bar{\partial}_A \Phi = 0.$$

Therefore, the set HC of holomorphic (D_A, Φ) pairs corresponds bijectively to the set Hig of Higgs bundles $(\bar{\partial}, \Phi)$.

The complex gauge group $\mathcal{G}^c$ acts on Hig naturally. Since $\mathcal{G} \subset \mathcal{G}^c$, $\mathcal{G}$ acts on the Higgs fields. Hence $\mathcal{G}$ acts on HC. The group $\mathcal{G}^c$ is much larger than $\mathcal{G}$; hence, one can expect the space $\mathrm{HC}/\mathcal{G}$ to be much larger than the space $\mathrm{Hig}/\mathcal{G}^c$. The key issue of this analysis is to establish an equivalence between the stable Higgs bundles in $\mathrm{Hig}/\mathcal{G}^c$ and the irreducible pairs in $\mathrm{HC}/\mathcal{G}$ satisfying Hitchin's self-duality equation.

A holomorphic subbundle L of $(V, \bar{\partial})$ is Φ-invariant if

$$\Phi(L) \subseteq L \otimes K.$$

A Higgs bundle $(\bar{\partial}, \Phi)$ is *stable* (*semi-stable*) if L being Φ-invariant implies

$$\deg(L) < (\leq) \frac{1}{2} \deg(V).$$

A Higgs bundle is *poly-stable* if it is a direct sum of stable Higgs bundles of the same degree. Denote by Hig^s and Hig^{ss} the space of stable and poly-stable Higgs bundles on X. The action of $\mathcal{G}^c$ preserves Hig^s and Hig^{ss}; hence, one may define the moduli spaces

$$\mathcal{H}^s = \mathrm{Hig}^s/\mathcal{G}^c, \quad \mathcal{H}^{ss} = \mathrm{Hig}^{ss}/\mathcal{G}^c.$$

The space $\mathcal{H}^{ss}$ is a coarse moduli space parameterizing $\mathcal{G}^c$-equivalence classes of poly-stable Higgs bundles while $\mathcal{H}^s$ is a fine moduli space of stable Higgs bundles [9,12]. Denote by $\mathcal{H}_0^s, \mathcal{H}_1^s, \mathcal{H}_0^{ss}, \mathcal{H}_1^{ss}$ the components of stable and poly-stable Higgs bundles associated with $c_1(P^c)$ being 0 and 1, respectively.

A pair (D_A, Φ) is *irreducible* if the connection $D_A + \Phi + \Phi^*$ is irreducible. A pair (D_A, Φ) is *semi-simple* if $D_A + \Phi + \Phi^*$ is a direct sum of irreducible connections of the same degree. A holomorphic pair (D_A, Φ) is called *self-dual* if it satisfies Hitchin's self-duality equation [9]:

$$F(D_A) + [\Phi, \Phi^*] = \frac{1}{2} tr(F(D_0))I.$$

Let YM^s and YM^{ss} denote the spaces of irreducible and semi-simple self-dual pairs. The action of $\mathcal{G}$ preserves both the properties of irreducibility and self-duality; hence, one may define the moduli spaces

$$\mathcal{YM}^s = YM^s/\mathcal{G}, \quad \mathcal{YM}^{ss} = YM^{ss}/\mathcal{G}.$$

Hitchin showed that each $\mathcal{G}^c$ orbit in $\mathcal{H}^s$ contains a Higgs bundle $(\bar\partial, \Phi)$ such that its corresponding pair (D_A, Φ) is a self-dual pair with

$$\bar\partial_A = \bar\partial.$$

Moreover the Higgs bundle $(\bar\partial, \Phi)$ is unique up to $\mathcal{G}$-equivalence. In other words, the two moduli spaces $\mathcal{H}^s$ and $\mathcal{YM}^s$ are diffeomorphic. Furthermore, given any self-dual pair in YM^s, the connection

$$D = D_A + \Phi + \Phi^*$$

is flat and irreducible for $c_1 = 0$ and descends to a flat $\mathrm{PSL}(2, \mathbb{C})$ connection for $c_1 = 1$. From now on, we shall always assume V to have a holomorphic structure and write V_0 for $\bar\partial_0$ and (V, Φ) for a poly-stable Higgs bundle instead of $(\bar\partial, \Phi)$. We call the connection D, so constructed from a Higgs bundle (V, Φ), the connection associated with (V, Φ).

The 2-torsion subgroup $J_2(X)$ acts on $\mathcal{H}^{ss}$ by

$$L.(V, \Phi) = (L \otimes V, \Phi).$$

Theorem 4.1 **(**Hitchin [9,12]**).**

$$\begin{aligned} \mathcal{H}^s_c &\cong M^s_c / \mathrm{PSL}(2, \mathbb{C}), \\ \mathcal{H}^{ss}_c &\cong M^{ss}_c / \mathrm{PSL}(2, \mathbb{C}), \\ J_2(X)\backslash\mathcal{H}^s_c &\cong PM^s_c / \mathrm{PSL}(2, \mathbb{C}), \\ J_2(X)\backslash\mathcal{H}^{ss}_c &\cong PM^{ss}_c / \mathrm{PSL}(2, \mathbb{C}). \end{aligned}$$

Denote by g the identification maps of these spaces.

It is straightforward to generalize the notion of stability, semi-stability and poly-stability to Higgs bundles (V, Φ) with $c_1(V)$ equal to any integer. Define $\mathcal{H}^{ss}_c$ to be the moduli space of $\mathcal{G}^c$-equivalence classes of poly-stable Higgs bundle (V, Φ) with $c_1(V) = c$.

Let (L_d, D_L) be a holomorphic line bundle of degree d with a connection D_L. The line bundle L_d defines an isomorphism between the space of holomorphic bundles of a fixed first Chern class c with the space of holomorphic bundles with first Chern class $c + 2d$:

$$V_c \xmapsto{L_d \otimes} V_{c+2d}.$$

Moreover if V has a connection D, then the projective bundles $(P(V), D)$ and $(P(L_d \otimes V), D_L \otimes D)$ are isomorphic. Define

$$U = L_d \otimes V,$$

where $c_1(V)$ is either 0 or 1. Then

$$U \otimes U^* = (L_d \otimes V) \otimes (L_d \otimes V)^* = V \otimes V^*,$$

and

$$\mathrm{End}_0\, U \otimes K = \mathrm{End}_0\, V \otimes K.$$

Hence L_d defines an isomorphism

$$(V, \Phi) \xmapsto{L_d} (L_d \otimes V, \Phi)$$

which is $\mathcal{G}^c$-equivariant, hence, defines an isomorphism from $\mathcal{H}^{ss}_c$ to $\mathcal{H}^{ss}_{c+2d}$.

Corollary 4.2. *The components* $\mathcal{H}_c^{ss}$ *and* $J_2(X)\backslash\mathcal{H}_c^{ss}$ *are homeomorphic to* $M_{c'}^{ss}/\operatorname{PSL}(2,\mathbb{C})$ *and* $PM_{c'}^{ss}/\operatorname{PSL}(2,\mathbb{C})$*, respectively if* $c \equiv c' \mod 2$.

Fix the Hermitian metric $\tilde{h} = \pi^*(h)$ on $\tilde{X}$.

Definition 4.3. *Construct the above moduli spaces on the double cover* $\tilde{X}$ *and denote these objects by a* $\tilde{}$*. For example,* $\tilde{h} = \pi^*(h)$ *is the pull-back Hermitian metric on* $\tilde{V}$ *and* $\tilde{\mathcal{H}}^{ss}$ *is the coarse moduli space of poly-stable Higgs bundles on* $\tilde{X}$.

The involution τ induces a pull-back action τ^* on $\tilde{\mathcal{H}}^{ss}$.

Proposition 4.4. *The involution* τ^* *commutes with* $\tilde{g}$.

Proof. Since τ preserves $\tilde{h}$ and the complex structure on $\tilde{X}$, all the the operations involved in the identification map $\tilde{g}$ commute with τ^*.

One can see this locally by choosing an acyclic cover $\{\tilde{U}_i, \tilde{V}_i\}$ on $\tilde{X}$ symmetric with respect to τ in the sense that

$$\begin{cases} \tau(\tilde{U}_i) &= \tilde{V}_i \\ \tau(\tilde{V}_i) &= \tilde{U}_i \\ \tilde{U}_i \cap \tilde{V}_i &= \emptyset. \end{cases}$$

Such a cover is possible because τ does not fix any point and preserves the complex structure on $\tilde{X}$. □

4.2. The real case. Now we turn our attention to the subsets of $\mathcal{H}^s$ and $\mathcal{H}^{ss}$ that correspond to $N^s/\operatorname{PGL}(2,\mathbb{R})$ and $N^{ss}/\operatorname{PGL}(2,\mathbb{R})$.

We say a Higgs bundle (V, Φ) satisfies the *reality condition* or is a *real Higgs bundle* [9] if

1. There is a holomorphic line bundle L such that

$$V = L \oplus (L^{-1} \otimes \det(V)),$$

2. $\Phi = \Phi_1 \oplus \Phi_2$, where

$$\Phi_1 : L \longrightarrow L^{-1} \otimes \det(V) \otimes K,$$

$$\Phi_2 : L^{-1} \otimes \det(V) \longrightarrow L \otimes K,$$

i.e. Φ is of the form:

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},$$

where b and c are holomorphic sections of the bundles $L^2 \otimes K \otimes \det(V)^{-1}$ and $L^{-2} \otimes K \otimes \det(V)$, respectively.

For such a Higgs bundle (V, Φ), the line bundle L inherits a Hermitian metric h_1 from h. Let D be the connection associated to (V, Φ). The metric h_1 defines a bundle isomorphism between $\bar{L}$ and L^{-1}. This induces an anti-holomorphic linear transformation

$$f : V \longrightarrow V \otimes \det(V)^{-1},$$

$$f(u_1, u_2) = (\bar{u}_2, \bar{u}_1).$$

In addition, D commutes with f. Thus, D is a flat connection on the projective subbundle $PE \subset P(V)$ fixed by f. Moreover, (PE, D) is a flat $\mathrm{PSL}(2, \mathbb{R})$-bundle and the Euler class of PE equals $2\deg(L) - \deg(V)$.

Let $\mathbb{R}\mathcal{H}_e^{ss}$ be the subset of $\mathcal{H}^{ss}$ of poly-stable real Higgs bundle with Euler class e and $\mathbb{R}\mathcal{H}_e^{s}$ the subset of $\mathbb{R}\mathcal{H}_e^{ss}$ of stable real Higgs bundles.

Theorem 4.5 (Hitchin [9]**).** *The moduli space* $\mathbb{R}\mathcal{H}_e^{ss}$ *is homeomorphic to* $N_e^{ss}/\mathrm{PSL}(2, \mathbb{R})$, *and*

$$J_2(X)\backslash\mathbb{R}\mathcal{H}_e^{ss} \cong PN_e^{ss}/\mathrm{PSL}(2, \mathbb{R}).$$

The subspaces $\mathbb{R}\mathcal{H}_e^{s}$ *and* $J_2(X)\backslash\mathbb{R}\mathcal{H}_e^{s}$ *are diffeomorphic to* $N_e^{s}/\mathrm{PSL}(2, \mathbb{R})$ *and* $PN_e^{s}/\mathrm{PSL}(2, \mathbb{R})$, *respectively.*

Corollary 4.6. *Tensoring with a line bundle* L_d *of degree* d *gives a one-to-one correspondence between the real Higgs bundles in* $\mathcal{H}_c^{ss}$ *with Euler class* e *and the real Higgs bundles in* $\mathcal{H}_{c+2d}^{ss}$ *with Euler class* e.

5. Stability

This section is a study of stability criteria of real Higgs bundles.

Proposition 5.1. *Suppose*

$$V = L_1 \oplus L_2, \text{ with } d = \deg(L_1) = \deg(L_2).$$

Then the bundles L_1 *and* L_2 *are the only two holomorphic subbundles of* V *of degree* d *if and only if* $L_1 \neq L_2$.

Proof. If $L_1 = L_2$, then $ts \oplus (1-t)s$ generates a line subbundle of V for all $t \in [0, 1]$, where s is a meromorphic section of L_1 [8].

Suppose $L_1 \neq L_2$. Let $H \subset V$ be a holomorphic line bundle of degree d. Then H corresponds to a holomorphic section φ of the bundle

$$H^{-1} \otimes V = H^{-1} \otimes L_1 \oplus H^{-1} \otimes L_2$$

such that φ has no zero. Thus

$$\varphi = \varphi_1 \oplus \varphi_2,$$

where φ_1 is a section of $H^{-1} \otimes L_1$ and φ_2 of $H^{-1} \otimes L_2$. Since φ has no poles, neither do φ_1 and φ_2. However

$$\deg(H^{-1} \otimes L_1) = \deg(H^{-1} \otimes L_2) = 0,$$

so φ_1 is either identically zero or has no zero. The same is true with φ_2.
If $\varphi_1 \equiv 0$, then φ_2 has no zero and $H = L_2$.
If $\varphi_2 \equiv 0$, then φ_1 has no zero and $H = L_1$.
If neither φ_1, φ_2 has any zero, then $H = L_1$ and $H = L_2$. This is the case of $L_1 = L_2$.
□

Proposition 5.2. *Suppose (V, Φ) is a Higgs bundle on X and $\tilde{V} = \pi^*(V)$. In addition, suppose*

$$\tilde{V} = \tilde{L}_1 \oplus \tilde{L}_2$$

with

$$\begin{cases} \deg(\tilde{L}_1) = \deg(\tilde{L}_2) = d \\ \tau^*(\tilde{L}_1) \ = \tilde{L}_2 \\ \tau^*(\tilde{L}_2) \ = \tilde{L}_1 \\ \tilde{L}_1 \quad \neq \tilde{L}_2. \end{cases}$$

Then (V, Φ) is stable.

Proof. Suppose $H \subset V$. Then $\tilde{H} = \pi^*(H) \subset \tilde{V}$. Hence, by Proposition 5.1, $\tilde{H} = \tilde{L}_1$ or $\tilde{H} = \tilde{L}_2$ or $\deg(\tilde{H}) < d$. On the other hand, since $\tau^*(\tilde{H}) = \tilde{H}$, it must be the case that $\deg(\tilde{H}) < d$. This implies V is a stable holomorphic bundle. Hence (V, Φ) is stable for any Φ. □

6. The Flat PGL(2, $\mathbb{R}$) Structures

Suppose (V, Φ) is a Higgs bundle on X. Then (V, Φ) pulls back to

$$(\tilde{V}, \tilde{\Phi}) = \pi^*(V, \Phi).$$

Proposition 4.4 indicates that one needs to determine the set of stable Higgs bundles of the form (V, Φ) on X such that $\pi^*(V, \Phi) \in \mathbb{R}\tilde{\mathcal{H}}_0^{ss}$ and $\det(V)$ is $\det(V_0)$.

The pull-back $\tilde{V}_0 = \pi^*(V_0)$ is a holomorphic bundle on $\tilde{X}$ and $\det(\tilde{V}_0)$ is a line bundle on $\tilde{X}$. Suppose $\deg(V_0) = 1$. Then $\deg(\tilde{V}_0) = 2$. By Proposition 3.2, there exists a line bundle $\tilde{L}_1$ and a 2-torsion line bundle $\tilde{T}$ such that

$$\tilde{L}_1^2 = \det(\tilde{V}_0), \quad \tau^*(\tilde{L}_1) = \tilde{L}_1 \otimes \tilde{T}.$$

Definition 6.1.

$$\begin{aligned} \mathcal{Q}_0 &= \{(\tilde{L}, \tilde{b}) : \tilde{L} \in \mathcal{P}, \tilde{L}^2 \neq 1, \tilde{b} \in \mathrm{H}^0(\tilde{X}, \tilde{L}^2\tilde{K})\}, \\ \mathcal{Q}_1 &= \{(\tilde{L} \otimes \tilde{L}_1, \tilde{b}) : \tilde{L} \in \mathcal{P}, \tilde{b} \in \mathrm{H}^0(\tilde{X}, \tilde{L}^2 \otimes \tilde{T} \otimes \tilde{K})\}, \\ \mathcal{Q} &= \mathcal{Q}_0 \cup \mathcal{Q}_1. \end{aligned}$$

The group $J_2(\tilde{X}) \cap \mathcal{P}$ acts on $\mathcal{Q}$:

$$\begin{aligned} (J_2(\tilde{X}) \cap \mathcal{P}) \times \mathcal{Q} &\longrightarrow \mathcal{Q}, \\ (\tilde{L}', (\tilde{L}, \tilde{b})) &\longrightarrow (\tilde{L}' \otimes \tilde{L}, \tilde{b}). \end{aligned}$$

The quotients are denoted by $P\mathcal{Q}_0$, $P\mathcal{Q}_1$, $P\mathcal{Q}$, respectively.

Proposition 6.2. *The spaces $PW_0'/\,\mathrm{PGL}(2, \mathbb{R})$ and $PW_1'/\,\mathrm{PGL}(2, \mathbb{R})$ are diffeomorphic to $P\mathcal{Q}_0$ and $P\mathcal{Q}_1$, respectively.*

Proof. Case 1: $w_2 = 0$. Let $\sigma \in W_0' / \mathrm{PGL}(2, \mathbb{R})$. Then σ corresponds to an element in $\mathcal{H}_0^s$. Let

$$\tilde{\sigma} = \pi^*(\sigma).$$

By Proposition 2.4 and Theorem 4.5, $\tilde{\sigma}$ is an $\mathrm{SL}(2, \mathbb{R})$ representation and corresponds to an element in $\mathbb{R}\tilde{\mathcal{H}}_0^{ss}$, hence, to a poly-stable Higgs bundle $(\tilde{V}, \tilde{\Phi})$ such that:

$$\pi^*(V) = \tilde{V} = \tilde{L} \oplus \tilde{L}^{-1},$$
$$\pi^*(\Phi) = \tilde{\Phi} = \tilde{\Phi}_1 \oplus \tilde{\Phi}_2 = \begin{pmatrix} 0 & \tilde{b} \\ \tilde{c} & 0 \end{pmatrix}$$

with

$$\deg(\tilde{L}) = 0.$$

Since τ preserves the degree of any divisor, τ^* preserves the degree of $\tilde{L}$. Suppose $\tilde{L}^2 \neq 1$. By Proposition 5.1, either

$$\begin{cases} \tau^*(\tilde{L}) &= \tilde{L} \\ \tau^*(\tilde{L}^{-1}) &= \tilde{L}^{-1} \end{cases} \quad \text{or} \quad \begin{cases} \tau^*(\tilde{L}) &= \tilde{L}^{-1} \\ \tau^*(\tilde{L}^{-1}) &= \tilde{L}. \end{cases}$$

This implies, after normalizing, the following dichotomy:

1.
$$\begin{cases} \tau^*(\tilde{L}) = \tilde{L}, & \tau^*(\tilde{L}^{-1}) = \tilde{L}^{-1}, \\ \tau^*(\tilde{\Phi}_1) = \tilde{\Phi}_1, & \tau^*(\tilde{\Phi}_2) = \tilde{\Phi}_2, \\ \tau^*(\tilde{b}) = \tilde{b}, & \tau^*(\tilde{c}) = \tilde{c}; \end{cases}$$

2.
$$\begin{cases} \tau^*(\tilde{L}) = \tilde{L}^{-1}, & \tau^*(\tilde{L}^{-1}) = \tilde{L}, \\ \tau^*(\tilde{\Phi}_1) = \tilde{\Phi}_2, & \tau^*(\tilde{\Phi}_2) = \tilde{\Phi}_1, \\ \tau^*(\tilde{b}) = \tilde{c}, & \tau^*(\tilde{c}) = \tilde{b}. \end{cases}$$

Let $\tilde{E} \subset \tilde{V}$ be the flat $\mathrm{SL}(2, \mathbb{R})$-bundle fixed by the anti-holomorphic map $\tilde{f}$. Let $\tilde{D}$ be the connection associated with $(\tilde{V}, \tilde{\Phi})$.

With the solutions to Eq. 1, τ^* preserves orientations on $\tilde{E}$; therefore, the pair $(\tilde{E}, \tilde{D})$ descends to a flat $\mathrm{SL}(2, \mathbb{R})$-bundle (E, D) $(w_1(E) = 0)$ of X. Note $\tilde{b} \in \mathrm{H}^0(\tilde{X}, \tilde{L}^2 \otimes K)$ and $\tilde{c} \in \mathrm{H}^0(\tilde{X}, \tilde{L}^{-2} \otimes K)$ with $\tilde{L} \in \mathcal{S}$. Alternatively, $(\tilde{V}, \tilde{\Phi})$ is a lift of a pair (V, Φ) that satisfies the reality condition. Note

$$\pi^*(V, \Phi) = \pi^*(V \otimes T_\eta, \Phi).$$

Hence the descent from $(\tilde{V}, \tilde{\Phi})$ is not unique. However these two Higgs bundles differ by a 2-torsion; hence, the projectivized bundles are the same.

With the solutions to Eq. 2, τ^* reverses orientations on $\tilde{E}$. Hence the pair $(\tilde{E}, \tilde{D})$ descends to a flat $\mathrm{SL}_i(2, \mathbb{R})$-bundle (E, D) on X. The two equations on $\tilde{L}, \tilde{L}^{-1}$ are precisely the condition for $\tilde{L}$ to be in $\mathcal{P}$.

The group $\mathcal{P}$ has four connected components. Let $L_\eta \in J(X)$ such that $L_\eta^2 = T_\eta$. Let $\tilde{L}_\eta = \pi^*(L_\eta)$. Then

$$\tilde{L}_\eta^2 = \pi^*(L_\eta)^2 = \pi^*(L_\eta^2) = \pi^*(T_\eta) = 1.$$

In other words, $\tilde{L}_\eta$ is a 2-torsion in $\mathcal{S}$, hence, it is also in $\mathcal{P}$.

Suppose V is a rank-2 holomorphic bundle on X with $\det(V) = 1$ such that $\tilde{V} = \pi^*(V) = \tilde{L} \oplus \tilde{L}^{-1}$, where $\tilde{L} \in \mathcal{P}$. Then $\tilde{L} \otimes \tilde{L}_\eta \in \mathcal{P}$ and

$$\pi^*(V \otimes L_\eta) = \tilde{V} \otimes \tilde{L}_\eta = (\tilde{L} \otimes \tilde{L}_\eta) \oplus (\tilde{L} \otimes \tilde{L}_\eta)^{-1}.$$

However,

$$\det(V \otimes L_\eta) = T_\eta \neq 1.$$

Therefore if $\tilde{L} \in \mathcal{P}$ and

$$\tilde{V} = \pi^*(V) = \tilde{L} \oplus \tilde{L}^{-1},$$

then either $\det(V) = 1$ or $\det(V) = T_\eta$. Since $\det(V)$ cannot jump on connected components of $\mathcal{P}$, it must be the case that only two components of $\mathcal{P}$ induce vector bundles V on X with $\det(V) = 1$. Denote these two components $\mathcal{P}_0'$. The components $\mathcal{P} \backslash \mathcal{P}_0'$ will induce bundles V with determinant T_η. Hence only the Higgs bundles induced by $\mathcal{P}_0'$ correspond to $\mathrm{SL}_i(2, \mathbb{R})$ representations.

Remark 6.3. The points in $\mathcal{Q}_0$ correspond to points in the space of $\mathrm{SL}_{\pm i}(2, \mathbb{R})$ representation classes.

Let

$$\mathcal{Q}_0' = \{(\tilde{L}, \tilde{b}) : \tilde{L} \in \mathcal{P}_0', \tilde{L}^2 \neq 1, \tilde{b} \in \mathrm{H}^0(\tilde{X}, \tilde{L}^2\tilde{K})\}.$$

By Corollary 2.11, this construction describes a 2-to-1 map

$$\Pi : W_0' / \mathrm{PGL}(2, \mathbb{R}) \longrightarrow \mathcal{Q}_0'.$$

Note the 2-torsions in $\mathcal{P}_0'$ are excluded because they correspond to the reducible representation classes $[\sigma]$ with $\sigma(a_0^2) = \pm I$, hence, are not in W_0' by definition.

To show Π is onto, let $(\tilde{L}, \tilde{b}) \in \mathcal{Q}_0'$ and

$$\tilde{V} = \tilde{L} \oplus \tilde{L}^{-1},$$

$$\tilde{\Phi} = \begin{pmatrix} 0 & \tilde{b} \\ \tau^*(\tilde{b}) & 0 \end{pmatrix}.$$

The involution τ^* on $\tilde{V}$ preserves the subspace $\tilde{E} \subset \tilde{V}$ but reverses orientations on $\tilde{E}$. Let $< \tau^* >$ be the order two group generated by τ^* and define the quotients

$$V = \tilde{V} / < \tau^* >,$$
$$\Phi = \tilde{\Phi} / < \tau^* >,$$
$$E = \tilde{E} / < \tau^* > .$$

Since $\tilde{L} \neq \tilde{L}^{-1}$, by Proposition 5.2, (V, Φ) is a stable Higgs bundle over X. Moreover $(\tilde{V}, \tilde{\Phi})$ and $\tilde{E}$ are pull-backs of (V, Φ) and E by π^*, respectively. Let $\tilde{D}$ be the connection associated with $(\tilde{V}, \tilde{\Phi})$. Then $(\tilde{E}, \tilde{D})$ is a flat $\mathrm{SL}(2, \mathbb{R})$-bundle and τ^* reverses orientations on $\tilde{E}$. Hence (E, D) is a flat $\mathrm{SL}_i(2, \mathbb{R})$-bundle on X. Hence the map Π is onto. Note the fibre of Π at the point $(\tilde{V}, \tilde{\Phi})$ consists of the two points (V, Φ) and $(V \otimes T_\eta, \Phi)$.

Case 2: $w_2 = 1$. The proof is similar to the proof of *Case 1*. A representation class $\sigma \in W_1'/\,\mathrm{PGL}(2, \mathbb{R})$ corresponds to an element in $\mathcal{H}_1^s$. Hence

$$\tilde{\sigma} = \pi^*(\sigma)$$

is an $\mathrm{SL}(2, \mathbb{R})$ representation. By Proposition 4.5 and Corollary 4.6, $\tilde{\sigma}$ corresponds to a poly-stable Higgs bundle $(\tilde{V}, \tilde{\Phi})$ such that:

$$\pi^*(V) = \tilde{V} = \tilde{L}_1 \otimes \tilde{V}_1 = \tilde{L}_1 \otimes \tilde{L} \oplus \tilde{L}_1 \otimes \tilde{L}^{-1},$$
$$\pi^*(\Phi) = \tilde{\Phi} = \tilde{\Phi}_1 \oplus \tilde{\Phi}_2 = \begin{pmatrix} 0 & \tilde{b} \\ \tilde{c} & 0 \end{pmatrix},$$

with $\deg(\tilde{L}) = 0$. Suppose

$$\tilde{L}_1 \otimes \tilde{L} \neq \tilde{L}_1 \otimes \tilde{L}^{-1}.$$

By Proposition 5.1, either

$$\begin{cases} \tau^*(\tilde{L}_1 \otimes \tilde{L}) & = \tilde{L}_1 \otimes \tilde{L} \\ \tau^*(\tilde{L}_1 \otimes \tilde{L}^{-1}) & = \tilde{L}_1 \otimes \tilde{L}^{-1} \end{cases} \quad \text{or} \quad \begin{cases} \tau^*(\tilde{L}_1 \otimes \tilde{L}) & = \tilde{L}_1 \otimes \tilde{L}^{-1} \\ \tau^*(\tilde{L}_1 \otimes \tilde{L}^{-1}) & = \tilde{L}_1 \otimes \tilde{L}. \end{cases}$$

This implies, after normalizing, the following dichotomy:

1.
$$\begin{cases} \tau^*(\tilde{L}_1 \otimes \tilde{L}) = \tilde{L}_1 \otimes \tilde{L}, & \tau^*(\tilde{L}_1 \otimes \tilde{L}^{-1}) = \tilde{L}_1 \otimes \tilde{L}^{-1}, \\ \tau^*(\tilde{\Phi}_1) \quad = \tilde{\Phi}_1, & \tau^*(\tilde{\Phi}_2) \quad = \tilde{\Phi}_2, \\ \tau^*(\tilde{b}) \quad = \tilde{b}, & \tau^*(\tilde{c}) \quad = \tilde{c}; \end{cases}$$

2.
$$\begin{cases} \tau^*(\tilde{L}_1 \otimes \tilde{L}) = \tilde{L}_1 \otimes \tilde{L}^{-1}, & \tau^*(\tilde{L}_1 \otimes \tilde{L}^{-1}) = \tilde{L}_1 \otimes \tilde{L}, \\ \tau^*(\tilde{\Phi}_1) \quad = \tilde{\Phi}_2, & \tau^*(\tilde{\Phi}_2) \quad = \tilde{\Phi}_1, \\ \tau^*(\tilde{b}) \quad = \tilde{c}, & \tau^*(\tilde{c}) \quad = \tilde{b}. \end{cases}$$

Equation 1 has no solution. Since $\tilde{V} = \pi^*(V)$, the equality

$$\tau^*(\tilde{L}_1 \otimes \tilde{L}) = \tilde{L}_1 \otimes \tilde{L}$$

would imply the existence of $L' \in V$ with

$$\tilde{L}_1 \otimes \tilde{L} = \pi^*(L').$$

Since $\deg(\tilde{L}_1 \otimes \tilde{L}) = 1$, the degree of L' would have been $\frac{1}{2}$. This is not possible.

With the solutions to Eq. 2, τ^* reverses orientations on the $\mathrm{SL}(2, \mathbb{R})$-bundle $\tilde{E} \subset \tilde{V}$. Let $\tilde{D}$ be the connection associated with $(\tilde{V}, \tilde{\Phi})$. Then the pair $(\tilde{E}, \tilde{D})$ descends to an

$\mathrm{SL}_i(2,\mathbb{R})$-bundle (E, D) on X. The bundle further descends to a flat projective bundle $(P(E), D)$ on X.

Since $\tilde{T}$ is a 2-torsion in $\mathcal{P}$,

$$\tau^*(\tilde{T}) = \tilde{T}^{-1} = \tilde{T}.$$

Hence $\tilde{T} \in \mathcal{S}$. Since $\mathcal{S}$ is an abelian variety, there exists $\tilde{L}_2 \in \mathcal{S}$ such that

$$\tilde{T} = \tilde{L}_2^2.$$

This implies

$$\begin{cases} \tau^*(\tilde{L}_2) \quad = \tilde{L}_2 \quad = \tilde{T} \otimes \tilde{L}_2^{-1} \\ \tau^*(\tilde{L}_2^{-1}) = \tilde{L}_2^{-1} = \tilde{T} \otimes \tilde{L}_2. \end{cases}$$

Hence for each $\tilde{L} \in \mathcal{P}$,

$$\begin{cases} \tau^*((\tilde{L} \otimes \tilde{L}_2) \otimes \tilde{L}_1) \quad = (\tilde{L} \otimes \tilde{L}_2)^{-1} \otimes \tilde{L}_1 \\ \tau^*((\tilde{L} \otimes \tilde{L}_2)^{-1} \otimes \tilde{L}_1) = (\tilde{L} \otimes \tilde{L}_2) \otimes \tilde{L}_1. \end{cases}$$

Hence the solutions to Eq. 2 give points in $\mathcal{Q}_1$. Note, by Proposition 3.2, $\tilde{T} \notin \mathcal{P}_0$. Thus there does not exist $\tilde{L} \in \mathcal{P}$ such that $\tilde{L}^2 \otimes \tilde{T} = 1$. This implies that there does not exist $\tilde{L} \in \mathcal{P}$ such that

$$\tilde{L} \otimes \tilde{L}_2 \otimes \tilde{L}_1 = (\tilde{L} \otimes \tilde{L}_2)^{-1} \otimes \tilde{L}_1.$$

This gives a 2-to-1 map

$$\Pi : W_1' / \mathrm{PGL}(2,\mathbb{R}) \longrightarrow \mathcal{Q}_1.$$

Similar to Case 1, there are only two components of $\mathcal{P}$ that induce vector bundles V such that

$$\pi^*(V) = \tilde{V} = \tilde{L}_1 \otimes (\tilde{L}_2 \otimes \tilde{L}) \oplus \tilde{L}_1 \otimes (\tilde{L}_2 \otimes \tilde{L})^{-1},$$

with $\det(V) = \det(V_0)$. Denote these two components $\mathcal{P}_1'$ and define

$$\mathcal{Q}_1' = \{(\tilde{L} \otimes \tilde{L}_1, \tilde{b}) : \tilde{L} \in \mathcal{P}_1', \tilde{b} \in \mathrm{H}^0(\tilde{X}, \tilde{L}^2 \otimes \tilde{T} \otimes \tilde{K})\}.$$

Let $(\tilde{L}, \tilde{b}) \in \mathcal{Q}_1'$ and

$$\tilde{V} = \tilde{L}_1 \otimes (\tilde{L}_2 \otimes \tilde{L}) \oplus \tilde{L}_1 \otimes (\tilde{L}_2 \otimes \tilde{L})^{-1},$$

$$\tilde{\Phi} = \begin{pmatrix} 0 & \tilde{b} \\ \tau^*(\tilde{b}) & 0 \end{pmatrix}.$$

The involution τ^* on $\tilde{V}$ preserves the subspace $\tilde{E} \subset \tilde{V}$ but reverses orientations on $\tilde{E}$. Let $< \tau^* >$ be the order two group generated by τ^* and define quotient sets

$$V = \tilde{V} / < \tau^* >,$$
$$\Phi = \tilde{\Phi} / < \tau^* >,$$
$$E = \tilde{E} / < \tau^* > .$$

By Proposition 5.2, (V, Φ) is a stable Higgs bundle over X. Moreover, $(\tilde{V}, \tilde{\Phi})$ and $\tilde{E}$ are pull-backs of (V, Φ) and E by π^*, respectively. Since $(\tilde{E}, \tilde{D})$ is an $\mathrm{SL}(2,\mathbb{R})$-bundle and

τ^* reverses orientations on $\tilde{E}$, (E, D) is an $\mathrm{SL}_i(2,\mathbb{R})$-bundle on X. The bundle (E, D) descends to a flat projective bundle $(P(E), D)$ ($\mathrm{PGL}(2,\mathbb{R})$-bundle) on X. Hence Π is onto. Finally, by Corollary 2.11,

$$P\mathcal{Q} = (J_2(\tilde{X}) \cap \mathcal{P})\backslash\mathcal{Q} = (J_2(X) \cap W)\backslash W'/\,\mathrm{PGL}(2,\mathbb{R}) = PW'/\,\mathrm{PGL}(2,\mathbb{R}). \qquad \square$$

Let $\mathcal{Q}' = \mathcal{Q}_0' \cup \mathcal{Q}_1'$. Proposition 6.2 actually provides an explicit identification of $W'/\,\mathrm{PGL}(2,\mathbb{R})$ with $\mathcal{Q}'$. This is stronger than needed to obtain Theorem 1.1. Since

$$P(\tilde{V} \otimes \tilde{L}') = P(\tilde{V})$$

for any line bundle $\tilde{L}'$, an alternative approach is to look at the equation

$$\tau^*(\tilde{V}, \tilde{\Phi}) = (\tilde{V} \otimes \tilde{L}', \tilde{\Phi}).$$

For the component $PW'/\,\mathrm{PGL}(2,\mathbb{R})$, this leads to the system of equations:

$$\begin{cases} \tau^*(\tilde{L}) = \tilde{L}^{-1} \otimes \tilde{L}', & \tau^*(\tilde{L}^{-1}) = \tilde{L} \otimes \tilde{L}' \\ \tau^*(\tilde{\Phi}_1) = \tilde{\Phi}_2, & \tau^*(\tilde{\Phi}_2) = \tilde{\Phi}_1, \\ \tau^*(\tilde{b}) = \tilde{c}, & \tau^*(\tilde{c}) = \tilde{b}. \end{cases}$$

The solutions to this system of equations correspond to the $\mathrm{GL}(2,\mathbb{C})$ connections that project down to flat $\mathrm{PGL}(2,\mathbb{R})$ connections.

The quotient $P\mathcal{P}$ is homeomorphic to a compact complex torus with complex dimension $g-1$. If $w_2 = 0$, then above each $[\tilde{L}] \in P\mathcal{P}\setminus\{[1]\}$ is the vector space $\mathrm{H}^0(\tilde{X}, \tilde{L}^2\tilde{K})$. By the Riemann-Roch formula,

$$h^0(\tilde{L}^2\tilde{K}) - h^0(\tilde{L}^{-2}) = 1 - (2g-1) + \deg(\tilde{L}^2\tilde{K}).$$

This implies

$$h^0(\tilde{L}^2\tilde{K}) = 1 - (2g-1) + [2(2g-1) - 2] = 2g - 2.$$

Hence the total dimension is

$$h^0(\tilde{L}^2\tilde{K}) + \dim(P\mathcal{P}) = 3g - 3.$$

Note if $\tilde{L}$ is a 2-torsion, then $h^0(\tilde{L}^2\tilde{K}) = 2g - 1$.

Suppose $w_2 = 1$. Again there is no $\tilde{L} \in \mathcal{P}$ such that $\tilde{L}^2 \otimes \tilde{T} = 1$. This implies that above any $[\tilde{L}] \in P\mathcal{P}$, the vector space $\mathrm{H}^0(\tilde{X}, \tilde{L}^2 \otimes \tilde{T} \otimes \tilde{K})$ is of dimension

$$h^0(\tilde{L}^2 \otimes \tilde{T} \otimes \tilde{K}) = 1 - (2g-1) + [2(2g-1) - 2] = 2g - 2.$$

Again, the total dimension is

$$h^0(\tilde{L}^2 \otimes \tilde{T} \otimes \tilde{K}) + \dim(P\mathcal{P}) = 3g - 3.$$

Finally, by Corollary 2.8, PW' is open and dense in PW. Therefore $PW'/\,\mathrm{PGL}(2,\mathbb{R})$ is open and dense in $PW/\,\mathrm{PGL}(2,\mathbb{R})$. This proves Theorem 1.1.

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