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The moduli of flat PU(p, p)-structures with large Toledo invariants

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Abstract. For a compact Riemann surface X of genus $g > 1$, $\mathrm{Hom}(\pi_1(X), \mathrm{PU}(p,q))/\mathrm{PU}(p,q)$ is the moduli space of flat $\mathrm{PU}(p,q)$-connections on X. There are two invariants, the Chern class c and the Toledo invariant τ associated with each element in the moduli. The Toledo invariant is bounded in the range $-2min(p,q)(g-1) \leq \tau \leq 2min(p,q)(g-1)$. This paper shows that the component, associated with a fixed $\tau > 2(max(p,q)-1)(g-1)$ (resp. $\tau < -2(max(p,q)-1)(g-1)$) and a fixed Chern class c, is connected (The restriction on τ implies $p = q$).

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1. Introduction and results

Let X be a smooth projective curve over $\mathbb{C}$ with genus $g > 1$. Let $\mathrm{PGL}(n,\mathbb{C})$ and $\mathrm{PU}(p,q)$ be $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(p,q)$ modulo their respective centers. The deformation space

$$\mathbb{C}\mathcal{N}_B = \mathrm{Hom}^+(\pi_1(X), \mathrm{PGL}(n,\mathbb{C}))/\mathrm{PGL}(n,\mathbb{C})$$

is the space of equivalence classes of semi-simple $\mathrm{PGL}(n,\mathbb{C})$-representations of the fundamental group $\pi_1(X)$. This is the $\mathrm{PGL}(n,\mathbb{C})$-Betti moduli space on X.

Since $\mathrm{PU}(p,q) \subset \mathrm{PGL}(n,\mathbb{C})$, $\mathbb{C}\mathcal{N}_B$ contains the space

$$\mathcal{N}_B(p,q) = \mathrm{Hom}^+(\pi_1(X), \mathrm{PU}(p,q))/\mathrm{PU}(p,q).$$

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The space $\mathcal{N}_B(p,q)$ will be referred to as the $\mathrm{PU}(p,q)$-Betti moduli space. One may assume $p \geq q > 0$ without loss of generality.

The Betti moduli spaces are of great interest in geometric topology and uniformization. Goldman analyzed $\mathcal{N}_B(1,1)$ and determined the number of its connected components [7]. A theorem of Corlette, Donaldson, Hitchin and Simpson gives a homeomorphism of $\mathbb{C}\mathcal{N}_B$ to two other moduli spaces—the $\mathrm{PGL}(n,\mathbb{C})$-de Rham and the $\mathrm{PGL}(n,\mathbb{C})$-Dolbeault moduli spaces, respectively [4,5,10,14]. The Dolbeault moduli spaces are moduli of semi-stable Higgs bundles. Hitchin subsequently considered $\mathcal{N}_B(1,1)$ from the Higgs bundle perspective and determined its topology [10]. The cases when the structure groups being $\mathrm{U}(p,1)$ and $\mathrm{PU}(2,1)$ are treated in [21,22]. Gothen obtained partial results for the structure groups $\mathrm{SU}(2,2)$ and $\mathrm{Sp}(4,\mathbb{R})$ [8]. Other related results have been obtained in [19,20].

Each element in $\mathcal{N}_B(p,q)$ is associated with a Chern class c and a Toledo invariant τ which is bounded as [6,17,18]

$$-2q(g-1) \leq \tau \leq 2q(g-1).$$

The main result presented here is the following:

Theorem 1.1. *The locus in $\mathcal{N}_B(p,q)$, associated with a fixed $\tau > 2(p-1)(g-1)$ (resp. $\tau < -2(p-1)(g-1)$) and a fixed Chern class c, is connected.*

Remark 1.2. The hypothesis $\tau > 2(p-1)(g-1)$ and the fact that $0 \leq \tau \leq 2q(g-1)$ imply $p = q$.

Section 2 reviews the homeomorphism between the $\mathrm{GL}(n,\mathbb{C})$-Betti space and the moduli space of Higgs bundles. Section 3 recalls the characterization of $\mathrm{U}(p,q)$-Higgs bundles and their Toledo invariant. Section 4 concerns the $\mathbb{C}^*$-action on the moduli space of $\mathrm{U}(p,p)$-Higgs bundles and introduces the locus of Binary hodge-bundles, a distinguished component of the $\mathbb{C}^*$-invariant locus. The main result of Sect. 4 (Proposition 4.6) implies that any point in the components (with $\tau > 2(p-1)(g-1)$) of the moduli can be deformed to a binary hodge bundle. In Sect. 5, we prove that the locus of binary hodge bundles is irreducible (Proposition 5.1). Theorem 1.1 follows from Propositions 4.6 and 5.1.

After the completion of this paper, S. Bradlow, O. Garcia-Prada and P. Gothen announced that all moduli spaces of flat $\mathrm{PU}(p,q)$-structures with fixed Chern class and Toledo invariant, are connected [3].

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2. The $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles

Let $\varGamma$ be the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \varGamma \longrightarrow \pi_1(X) \longrightarrow 1$$

as in [1,10]. Each $\rho \in \mathrm{Hom}(\varGamma, \mathrm{GL}(n,\mathbb{C}))$ acts on $\mathbb{C}^n$ via the standard representation of $\mathrm{GL}(n,\mathbb{C})$. The representation ρ is called reducible (resp. irreducible), if its action on $\mathbb{C}^n$ is reducible (resp. irreducible). A representation ρ is called semi-simple if it is a direct sum of irreducible representations.

Definition 2.1.

$$\mathbb{C}\mathcal{M}_B = \{\sigma \in \mathrm{Hom}(\varGamma, \mathrm{GL}(n,\mathbb{C})) : \sigma \textit{ is semi-simple}\}/\mathrm{GL}(n,\mathbb{C}).$$
$$\mathcal{M}_B(p,q) = \{\sigma \in \mathrm{Hom}(\varGamma, \mathrm{U}(p,q)) : \sigma \textit{ is semi-simple}\}/\mathrm{U}(p,q).$$

It is immediate that

$$\mathbb{C}\mathcal{N}_B = \mathbb{C}\mathcal{M}_B/\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$$
$$\mathcal{N}_B(p,q) = \mathcal{M}_B(p,q)/\mathrm{Hom}(\pi_1(X), \mathrm{U}(1)).$$

Therefore counting the components of $\mathcal{N}_B(p,q)$ is the same as counting the components of $\mathcal{M}_B(p,q)$.

Let E be a rank-$(p+q)$ complex vector bundle over X with $0 \le \deg(E) < p+q$. Denote by $\varOmega$ the canonical bundle on X. A holomorphic structure $\overline{\partial}$ on E induces holomorphic structures on the bundles $\mathrm{End}(E)$ and $\mathrm{End}(E) \otimes \varOmega$. A Higgs bundle is a pair $(E_{\overline{\partial}}, \varPhi)$, where $\overline{\partial}$ is a holomorphic structure on E and $\varPhi \in \mathrm{H}^0(X, \mathrm{End}(E_{\overline{\partial}}) \otimes \varOmega)$. Such a $\varPhi$ is called a Higgs field. We denote the holomorphic bundle $E_{\overline{\partial}}$ by V.

Define the slope of a vector bundle V to be

$$s(V) = \deg(V)/\mathrm{rank}(V).$$

For a fixed $\varPhi$, a holomorphic sub-bundle $W \subset V$ is said to be $\varPhi$-invariant if $\varPhi(W) \subset W \otimes \varOmega$. A Higgs bundle $(V, \varPhi)$ is stable (semi-stable) if $W \subset V$ being $\varPhi$-invariant implies

$$s(W) < (\le) s(V).$$

A Higgs bundle is called poly-stable if it is a direct sum of stable Higgs bundles of the same slope [10,15].

The Dolbeault moduli space $\mathbb{C}\mathcal{M}$ is the coarse moduli space of semi-stable rank-$(p+q)$ Higgs bundles on X [10–12,15]. The closed points of $\mathbb{C}\mathcal{M}$ parameterize the S-equivalent classes of semi-stable Higgs bundles. Moreover every S-equivalence class has a poly-stable representative.

3. The $\mathrm{U}(p, p)$-Higgs bundles

Definition 3.1. *Let $\mathcal{M}$ be the subset of $\mathbb{C}\mathcal{M}$ consisting of equivalent classes of Higgs bundles, whose poly-stable representative (V, Φ) satisfies the following two conditions:*

(1) V is a direct sum:

$$V = V_P \oplus V_Q,$$

where V_P, V_Q are of ranks p and q, respectively.

(2) The Higgs field decomposes into two maps:

$$\Phi_1 : V_P \longrightarrow V_Q \otimes \Omega,$$

$$\Phi_2 : V_Q \longrightarrow V_P \otimes \Omega.$$

Hence each $(V, \Phi) = (V_P \oplus V_Q, \Phi) \in \mathcal{M}$ is associated with two invariants, $d_P = \deg(V_P)$ and $d_Q = \deg(V_Q)$. The Toledo invariant is defined to be

$$\tau = 2\frac{\deg(V_P \otimes V_Q^*)}{p+q} = 2\frac{qd_P - pd_Q}{p+q},$$

and the Chern class is $d_P + d_Q$. The subset of $\mathcal{M}$, consisting of classes with fixed d_P and d_Q, is denoted by $\mathcal{M}_{(d_P, d_Q)}$.

Remark 3.2. When $p = q$, the labeling of the summands V_P and V_Q may seem ambiguous. However, we will introduce assumption (1) below, which implies that $d_P > d_Q$. Thus, when $p = q$, V_P is distinguished as the summand of larger degree.

Theorem 3.3.

(1) The moduli spaces $\mathbb{C}\mathcal{M}_B$ and $\mathbb{C}\mathcal{M}$ are homeomorphic.

(2) The reducible representations in $\mathbb{C}\mathcal{M}_B$ correspond to the poly(semi)-stable, but not stable, points.

(3) The subspace $\mathcal{M}_B$ is homeomorphic to $\mathcal{M}$.

(4) The space $\mathcal{M}_{(d_P, d_Q)}$ is homeomorphic to $\mathcal{M}_{(-d_P, -d_Q)}$.

Proof. The proof of (1) and (2) can be found in [4] (the main idea is present in [5,10] and the most general version of this celebrated result is in [14]). For (2) and (3), see [14,21]. See also the last few sections of [15] for general real forms. □

Part (3) characterizes isomorphism classes of semi-stable $\mathrm{U}(p, q)$-Higgs bundles as those, which are fixed under the involution $(V, \Phi) \mapsto (V, -\Phi)$ and the involution of V conjugating Φ to $-\Phi$ has eigenvalue -1 with multiplicity q. In particular, $\mathcal{M}$ is a closed subset of the quasi-projective variety $\mathbb{C}\mathcal{M}$, and we may endow $\mathcal{M}$ with the reduced induced subscheme structure.

By Theorem 3.3 (4), we assume, for the rest of the paper that

$$0 \le \tau \le 2q(g-1).$$

Lemma 3.4. *Suppose* $\tau > 2(p-1)(g-1)$*. Then* $(V_P \oplus V_Q, (\Phi_1, \Phi_2)) \in \mathcal{M}_{(d_P, d_Q)}$ *implies*

$$\Phi_1 : V_P \longrightarrow V_Q \otimes \Omega$$

is generically surjective.

Lemma 3.4 was first obtained for integer τ by Gothen [8].

Proof. Notice that the assumption $2(p-1)(g-1) < \tau \le 2q(g-1)$ implies $p = q$. Since $(V_P \oplus V_Q, \Phi)$ is semi-stable and τ is positive, $\Phi_1 \not\equiv 0$. Suppose Φ_1 is not generically surjective with a non-trivial kernel V_1. We want to produce a destabilizing Higgs subbundle, namely, $(V_P \oplus W_1 \otimes \Omega^{-1}, \Phi)$, where W_1 is as in the following canonical factorization of Φ_1:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{f_1} & V_P & \xrightarrow{f_2} & V_2 & \longrightarrow & 0 \\ & & & & \Phi_1 \big\downarrow & & \varphi \big\downarrow & & \\ 0 & \longleftarrow & W_2 & \xleftarrow{g_2} & V_Q \otimes \Omega & \xleftarrow{g_1} & W_1 & \longleftarrow & 0 \end{array} .$$

In the above diagram, the rows are exact, $\mathrm{rank}(V_2) = \mathrm{rank}(W_1)$ and φ has full rank at a generic point of X. Let $d_i = \deg(V_i)$, $r_i = \mathrm{rank}(V_i)$. Since V_1 is Φ-invariant, semi-stability implies

$$\frac{d_1}{r_1} \le \frac{d_P + d_Q}{2p}.$$

Since φ has full rank generically, $\deg(W_1) \ge d_2 = d_P - d_1$. Hence,

$$\begin{aligned} \deg(V_P \oplus W_1 \otimes \Omega^{-1}) &= d_P + \deg(W_1) - 2r_2(g-1) \\ &\ge 2d_P - \frac{r_1(d_P + d_Q)}{2p} - 2r_2(g-1). \end{aligned}$$

This implies

$$\begin{aligned} s(V_P \oplus W_1 \otimes \Omega^{-1}) - s(V) \ge \; & \frac{2d_P - \frac{r_1(d_P+d_Q)}{2p} - 2r_2(g-1)}{p + r_2} \\ & - \frac{d_P + d_Q}{2p}. \end{aligned}$$

This, together with the facts $r_1 + r_2 = p$ and $\tau = d_P - d_Q$, gives us

$$s(V_P \oplus W_1 \otimes \Omega^{-1}) - s(V) \ge \frac{\tau - 2r_2(g-1)}{p + r_2}.$$

Since $r_1 \ge 1$, we have $r_2 < p$. The assumption $\tau > 2(p-1)(g-1)$ then implies that $V_P \oplus W_1 \otimes \Omega^{-1}$ destabilizes V. □

For the rest of the paper, we assume

$$\tau > 2(p-1)(g-1). \tag{1}$$

4. The $\mathrm{U}(p,p)$-Hodge bundles

Definition 4.1 (See [15]). *A Hodge bundle on X is a Higgs bundle (E, Φ) of the following form:*

$$E = \bigoplus_{i=0}^{k} E^i$$

and $\Phi = (\phi_k, ..., \phi_1)$ with:

$$\phi_i : E^i \longrightarrow E^{i-1} \otimes \Omega.$$

The integer k is called the length of the Hodge bundle (E, Φ).

There is a $\mathbb{C}^*$-action on $\mathbb{C}\mathcal{M}$:

$$\begin{aligned} \mathbb{C}^* \times \mathbb{C}\mathcal{M} &\longrightarrow \mathbb{C}\mathcal{M} \\ (t, E, \Phi) &\mapsto (E, t\Phi). \end{aligned}$$

The space of equivalence classes of semi-stable Hodge bundles is the set of fixed points of this action (See Lemma 4.1 of [15]).

Remark 4.2. If (E, Φ) is a stable Higgs bundle, which is a fixed point of the $\mathbb{C}^*$-action, then it admits a *unique* decomposition of E, realizing it as a Hodge bundle (Lemma 4.1 in [15]). Hence, the decomposition is canonical if (E, Φ) is poly-stable.

A stable Hodge bundle (E, Φ) admits a unique realization as a $\mathrm{U}(p,q)$-Higgs bundle, for a unique pair of non-negative integers (p, q). Let $E = \oplus_{i=0}^{k} E^i$ be the unique decomposition of (E, Φ) and $\phi_i : E^i \to E^{i-1}$ the corresponding decomposition of the Higgs field. Let E^{odd} be the direct sum of the summands with odd index. We define E^{even} similarly. The stability of (E, Φ) implies, that there exists at most one automorphism f of E, up to a $\mathbb{C}^*$-factor, which conjugates Φ to $-\Phi$. Such an automorphism f is given by multiplying E^{odd} by $a \in \mathbb{C}^*$ and multiplying E^{even} by $-a$. Set $V_P = E^{\mathrm{odd}}$ and $V_Q = E^{\mathrm{even}}$. Then $(V_P \oplus V_Q, \Phi)$ is a $\mathrm{U}(p,q)$-Higgs bundle, with $p = \mathrm{rank}(V_P)$ and $q = \mathrm{rank}(V_Q)$. Recall our convention, that if $p = q$, then V_P is the summand of larger degree (Remark 3.2). This follows from assumption (1) by Lemma 3.4. Otherwise, $\deg(E^{\mathrm{even}}) > \deg(E^{\mathrm{odd}})$ and Lemma 3.4 implies that $\Phi : E^{\mathrm{even}} \to E^{\mathrm{odd}} \otimes \Omega$ is injective. This would contradict the fact that E^0 is in the kernel of Φ.

In Sect. 4.1 we describe the $\mathbb{C}^*$-action on the tangent space of a Hodge bundle. In Sect. 4.2 we introduce the locus of Binary Hodge bundles and characterize it in terms of the $\mathbb{C}^*$-action.

4.1. The infinitesimal $\mathbb{C}^*$-action

The infinitesimal deformations of a Higgs bundle (E,Φ) are calculated by the first cohomology of the complex $K_\bullet$ below

$$\mathrm{End}(E) \xrightarrow{ad_\Phi} \mathrm{End}(E)\otimes\Omega$$

(in degrees 0 and 1). When (E,Φ) is a Hodge bundle, $\mathrm{H}^1(K_\bullet)$ decomposes into weight spaces of the natural $\mathbb{C}^*$-action. Next, we analyze this decomposition. Let

$$\mu : \mathbb{C}^* \to \mathrm{Aut}(E^i) \quad \text{and}$$
$$\alpha : \mathbb{C}^* \to \mathrm{Aut}(\Omega)$$

be the representations with weights i and 1 respectively. We denote by μ also its natural extension to tensor products of the E^i and by α the action on $\mathrm{Hom}(E^i,E^j)\otimes\Omega$ via $1\otimes\alpha$. Set

$$\rho := \mu\cdot\alpha.$$

Then μ and ρ both have weight $j-i$ on $\mathrm{Hom}(E^i,E^j)$. On $\mathrm{Hom}(E^i,E^j\otimes\Omega)$, μ has weight $j-i$ and ρ has weight $j-i+1$. Observe that the Higgs field Φ, of a Hodge bundle, has weight -1 with respect to μ and it is ρ-invariant. Consequently, the differential of the complex $K_\bullet$ is ρ-invariant and the complex decomposes as a direct sum

$$K_\bullet = \bigoplus_{w=-k}^{k+1} K_\bullet^w, \tag{2}$$

where $K_\bullet^w$ is the complex

$$\bigoplus_{i=\max\{0,-w\}}^{\min\{k,k-w\}} \mathrm{Hom}(E^i,E^{i+w}) \xrightarrow{ad_\Phi} \bigoplus_{i=\max\{0,1-w\}}^{\min\{k,k-w+1\}} \mathrm{Hom}(E^i,E^{i+w-1}\otimes\Omega).$$

For example, $K_\bullet^{-k}$ is the complex supported in degree zero by the vector bundle $\mathrm{Hom}(E^k,E^0)$. The complex $K_\bullet^{1-k}$ is

$$\begin{array}{ccc} \mathrm{Hom}(E^{k-1},E^0)\bigoplus\mathrm{Hom}(E^k,E^1) & \xrightarrow{ad_\Phi} & \mathrm{Hom}(E^k,E^0\otimes\Omega), \\ (a,b) & \mapsto & \phi_1\circ b-(a\otimes 1)\circ\phi_k. \end{array} \tag{3}$$

Lemma 4.3. *The representation $\rho:\mathbb{C}^*\to\mathrm{Aut}(\mathrm{H}^1(K_\bullet))$ is the infinitesimal $\mathbb{C}^*$-action, on the tangent space at the fixed point (E,Φ), arising from the $\mathbb{C}^*$-action $(F,\varphi)\mapsto(F,t\varphi)$ on the moduli of Higgs bundles.*

Proof. Denote by $K_{\bullet,t}$ the complex corresponding to deformations of the Higgs pair $(E, t^{-1}\Phi)$. There are *two* natural isomorphisms of complexes

$$\begin{aligned} \mu_t &: \mathrm{H}^1(K_\bullet) \xrightarrow{\cong} \mathrm{H}^1(K_{\bullet,t}) \quad \text{and} \\ (1, t^{-1}) &: \mathrm{H}^1(K_\bullet) \xrightarrow{\cong} \mathrm{H}^1(K_{\bullet,t}) \end{aligned} \tag{4}$$

The first is the evaluation of the representation μ at t. The second is the identity on the vector bundle in degree 0 and multiplication by t^{-1} on the vector bundle in degree 1. The automorphism μ_t of E sends Φ to $t^{-1}\Phi$. Hence, (E, Φ) and $(E, t^{-1}\Phi)$ are equivalent Hodge bundles. In other words, Hodge bundles are $\mathbb{C}^*$-invariant. The isomorphism (4) is the natural identification of the two cohomological calculations of the same tangent space. The automorphism ρ_t of $K_\bullet$ is the composition

$$\rho_t \;=\; (1, t^{-1})^{-1} \circ \mu_t.$$

The lemma follows. □

Denote by $K_\bullet^{\text{even}}$ the sum of the even weight sub-complexes of $K_\bullet$.

Lemma 4.4. *Assume that (E, Φ) is stable. Then the infinitesimal $U(p,q)$-deformations of a Hodge bundle (E, Φ) are parameterized by $\mathrm{H}^1(K_\bullet^{\text{even}})$.*

Proof. This is the infinitesimal counterpart of the characterization of the $U(p,q)$ moduli space as the fixed point set of the involution

$$(E, \Phi) \;\mapsto\; (E, -\Phi).$$

□

4.2. Binary Hodge bundles

The $\mathbb{C}^*$-action on $\mathbb{C}\mathcal{M}$ preserves $\mathcal{M}_{(d_P,d_Q)}$ [15,21]. Let $\mathbb{C}\mathcal{H} \subset \mathbb{C}\mathcal{M}$ and $\mathcal{H}_{(d_P,d_Q)} \subset \mathcal{M}_{(d_P,d_Q)}$ be the corresponding sets of Hodge bundles.

Definition 4.5. *Suppose $(E, \Phi) = (V_P \oplus V_Q, (\Phi_1, \Phi_2)) \in \mathcal{M}_{(d_P,d_Q)}$. Then $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ is a binary Hodge bundle if $\Phi_2 \equiv 0$. That is a binary hodge bundle is a $\mathrm{U}(p,q)$-Hodge bundle of length 1. Denote by $\mathcal{B}_{(d_P,d_Q)}$ the set of binary Hodge bundles.*

$\mathcal{B}_{(d_P,d_Q)}$ is a closed subset of $\mathcal{M}_{(d_P,d_Q)}$. It may be endowed with the reduced induced subscheme structure. The closedness of $\mathcal{B}_{(d_P,d_Q)}$ follows from Lemma 4.8 below. Lemma 4.8 characterizes $\mathcal{B}_{(d_P,d_Q)}$ as the union of all connected components of the fixed locus of $\mathcal{M}_{(d_P,d_Q)}$, which are minimal with respect to a partial ordering by the weight invariant (5). We show later that $\mathcal{B}_{(d_P,d_Q)}$ is irreducible (Proposition 5.1).

The rest of this section is dedicated to proving the following proposition.

Proposition 4.6. *Under the assumption (1), $\mathcal{B}_{(d_P,d_Q)}$ intersects every connected component of $\mathcal{M}_{(d_P,d_Q)}$.*

The proposition follows from Lemmas 4.7 and 4.8. We will follow the general outline of Simpson's proof of the connectedness of the moduli space $\mathbb{C}\mathcal{M}$ (see [16]). Simpson introduced an algebraic version of a standard Morse theoretic technique. The space $\mathbb{C}\mathcal{M}$, as well as its subvariety $\mathcal{M}_{(d_P,d_Q)}$, are quasi-projective varieties. They fit into the following set-up. Let Y be a quasi-projective variety upon which $\mathbb{C}^*$ acts algebraically. Assume that L is a very ample line-bundle on Y with a linearization of the action. Then there is a finite dimensional invariant subspace $V \subset H^0(Y,L)$, giving rise to a $\mathbb{C}^*$-equivariant embedding $Y \hookrightarrow \mathbb{P}V^*$. Denote by $V = \oplus V_\alpha$ the decomposition into weight subspaces, so that $t \in \mathbb{C}^*$ acts by t^α on V_α. Let Z be the closure of Y in $\mathbb{P}V^*$. Denote by Z_β the intersection $Z \cap \mathbb{P}V^*_\beta$. The fixed locus of Z is the union of the loci Z_β. In particular, a connected component Z', of the fixed locus of Z, comes with an invariant; a weight

$$\beta(Z') \tag{5}$$

such that Z' is contained in $\mathbb{P}V^*_{\beta(Z')}$. Assume $y \in Y$ is not a fixed point. Then the closure of the $\mathbb{C}^*$-orbit of y in Z has two fixed points $y_0 := \lim_{t\to 0} ty$ and $y_\infty := \lim_{t\to\infty} ty$. Moreover, the invariants of y_0 and y_∞ satisfy the strict inequality $\beta(y_0) < \beta(y_\infty)$. Thus, connected components of the fixed locus are partially ordered. Moreover, if y is not a fixed point, taking the limit $\lim_{t\to 0} ty$ amounts to *flowing down to a lower connected component of the fixed locus of Z.*

Assume, furthermore, that $\lim_{t\to 0} tx$ exists in Y for all $x \in Y$. Then the process of flowing down can be used to study the connectedness of Y. More precisely, we have the following lemma, which is a trivial variation on Lemma 11.8 in [16]:

Lemma 4.7 (Lemma 11.8 in [16]). *Suppose Y is a quasi-projective variety, upon which $\mathbb{C}^*$ acts algebraically and $\lim_{t\to 0} tx$ exists in Y for all $x \in Y$ as above. Suppose $U \subset Y$ is a subset of the fixed point set of $\mathbb{C}^*$, and suppose that for any fixed point $x \notin U$, there exists $y \neq x$ such that $\lim_{t\to\infty} ty = x$. Then U intersects every connected component of Y.*

Proof. Suppose Y' is a connected component of Y not intersecting U. Let β be the smallest integer such that Y'_β is non-empty. Choose $x \in Y'_\beta$. By hypothesis, there exists $y \neq x$ in Y', such that $\lim_{t\to\infty} ty = x$. On the other hand, $z' := \lim_{t\to 0} ty$ is also in Y', say in Y'_α. Since y is not a fixed point, $\alpha < \beta$. This contradicts the minimality of β. □

It is known that $\lim_{t\to 0}(E, t\Phi)$ always exists in $\mathbb{C}\mathcal{M}$ and is a Hodge bundle [15]. The same holds for $\mathcal{M}_{(d_P,d_Q)}$ since $\mathcal{M}_{(d_P,d_Q)}$ is closed in $\mathbb{C}\mathcal{M}$

and is $\mathbb{C}^*$-invariant. Proposition 4.6 follows from Lemma 4.7, with $Y = \mathcal{M}_{(d_P,d_Q)}$ and $U = \mathcal{B}_{(d_P,d_Q)}$, and Lemma 4.8. The latter is the analogue of Lemma 11.9 of [16].

Lemma 4.8. *Suppose* $(E,\Phi) \in \mathcal{H}_{(d_P,d_Q)}$ *is poly-stable with length* $k > 1$. *Then there exists* $(F,\Psi) \in \mathcal{M}_{(d_P,d_Q)}$ *satisfying*

$$\lim_{t\to\infty}(F,t\Psi) \cong (E,\Phi) \quad \textit{and}$$
$$(F,\Psi) \ncong (E,\Phi).$$

Proof. Assume first that (E,Φ) is stable. Then it is a smooth point of $\mathcal{M}_{(d_P,d_Q)}$. Lemma 4.9 below implies that negative weights occur in the weight decomposition of the tangent space to $\mathcal{M}_{(d_P,d_Q)}$ at (E,Φ). Take a $\mathbb{C}^*$-orbit R in $\mathcal{M}_{(d_P,d_Q)}$, with (E,Φ) in its closure, such that the tangent line to R at (E,Φ) has negative weight. Then any point (F,Ψ) in R would satisfy the conditions of the Lemma.

Next, we reduce the proof to the stable case. Suppose $(E,\Phi) = (V_P \oplus V_Q, \Phi)$ is a poly-stable representative, of length $k > 1$, of an equivalence class in $\mathcal{H}_{(d_P,d_Q)}$. Then (E,Φ) is of the form $(V'_P \oplus V'_Q, \Phi') \oplus (V''_P \oplus V''_Q, \Phi'')$, where $(V'_P \oplus V'_Q, \Phi')$ is a stable Hodge bundle of length $k > 1$. Lemma 3.4 implies that Φ_1 is generically an isomorphism. Hence, the same holds for Φ'_1 and Φ''_1. Lemma 4.9 below implies the existence of a pair (F',Ψ'), not equivalent to $(V'_P \oplus V'_Q, \Phi')$, such that $\lim_{t\to\infty}(F',t\Psi') = (V'_P \oplus V'_Q, \Phi')$. Take $(F,\Psi) := (F',\Psi') \oplus (V''_P \oplus V''_Q, \Phi'')$. Then

$$\lim_{t\to\infty}(F,t\Psi) = \lim_{t\to\infty}(F',t\Psi') \oplus (V''_P \oplus V''_Q, \Phi'') = (E,\Phi).$$

Moreover, the graded objects $gr((E,\Phi))$ and $gr((F,\Psi))$ are not isomorphic. Hence, (E,Φ) and (F,Ψ) are not equivalent. □

Let (E,Φ) be a stable hodge bundle corresponding to a $\mathrm{U}(p,q)$-bundle $(V_P \oplus V_Q, \Phi)$. Since (E,Φ) is fixed by the $\mathbb{C}^*$-action, the tangent space $T_{(E,\Phi)}\mathcal{M}_{(d_P,d_Q)}$ is a representation of $\mathbb{C}^*$. Denote by $[T_{(E,\Phi)}\mathcal{M}_{(d_P,d_Q)}]^-$ the sum of negative weight spaces of this representation. Lemma 4.9 below is used in the proof of Lemma 4.8.

Lemma 4.9. *Let* (E,Φ) *be a stable hodge bundle corresponding to a* $\mathrm{U}(p,$ $p)$*-bundle* $(V_P \oplus V_Q, \Phi)$. *Assume that* $\Phi_1 : V_P \to V_Q \otimes \Omega$ *is an injective homomorphism. Assume further that the length* k *of its Hodge decomposition is* ≥ 2. *Then* $[T_{(E,\Phi)}\mathcal{M}_{(d_P,d_Q)}]^-$ *does not vanish.*

Proof. Note that we exclude the binary Hodge bundles which correspond to the case of $k = 1$. There are two cases, namely when k is even and when k is odd.

Case 1: $k > 2$ is odd

It suffices to prove that $\mathrm{H}^1(K_\bullet)^{1-k}$ does not vanish. Our assumptions imply that $\phi_{2i+1} : E^{2i+1} \to E^{2i}$ is generically an isomorphism. In particular, $r_{2i+1} = r_{2i}$, where r_i is the rank of E^i. Stability of the Hodge bundle (E,Φ) implies:

$$s(E^0 \oplus E^1) \quad < \quad s(E) \quad < \quad s(E^{k-1} \oplus E^k).$$

Lemma 4.10. *At least one of the two cases holds:*

$$s(E^0) < s(E^{k-1}) \quad \textit{or} \tag{6}$$

$$s(E^1) < s(E^k). \tag{7}$$

Proof. Assume otherwise. Then $s(E^0) \geq s(E^{k-1})$ and $s(E^1) \geq s(E^k)$. Using the fact that $r_0 = r_1$ and $r_{k-1} = r_k$, we get:

$$s(E^0 \oplus E^1) = \frac{s(E^0) + s(E^1)}{2} \geq \frac{s(E^{k-1}) + s(E^k)}{2} = s(E^{k-1} \oplus E^k).$$

This contradicts the stability of (E,Φ). □

We will use Lemma 4.10 to prove that $\mathrm{H}^1(K_\bullet)^{1-k}$ does not vanish. The $1-k$ weight space is equal to the first cohomology $\mathrm{H}^1(K_\bullet^{1-k})$ of the complex (3). $\mathrm{H}^i(K_\bullet^{1-k})$ vanishes for $i < 0$ and $i > 2$. We get the inequality

$$\dim \mathrm{H}^1(K_\bullet)^{1-k} \quad \geq \quad -\chi(K_\bullet^{1-k}).$$

Consequently, it suffices to prove that the Euler characteristic $\chi(K_\bullet^{1-k})$ is negative

$$\chi(\mathrm{Hom}(E^{k-1}, E^0)) + \chi(\mathrm{Hom}(E^k, E^1)) - \chi(\mathrm{Hom}(E^k, E^0 \otimes \Omega)) \quad < \quad 0. \tag{8}$$

Consider first the case $s(E^1) < s(E^k)$. The Euler characteristic $\chi(\mathrm{Hom}(E^k, E^1))$ is $r_1 r_k[s(E^1) - s(E^k) + 1 - g]$. It follows that $\chi(\mathrm{Hom}(E^k, E^1))$ is negative. Composition with ϕ_k

$$\mathrm{Hom}(E^{k-1}, E^0) \quad \xrightarrow{\phi_k} \quad \mathrm{Hom}(E^k, E^0 \otimes \Omega)$$

is generically an isomorphism by our assumption on Φ_1. Consequently, the difference of their Euler characteristics is negative

$$\chi(\mathrm{Hom}(E^{k-1}, E^0)) - \chi(\mathrm{Hom}(E^k, E^0 \otimes \Omega)) =$$
$$r_0 r_k[s(\mathrm{Hom}(E^{k-1}, E^0)) - s(\mathrm{Hom}(E^k, E^0 \otimes \Omega))] < 0.$$

Equation (8) follows.

The case $s(E^0) < s(E^{k-1})$ is similar. Use composition with ϕ_1 instead.

Case 2: k is even

Then the complex $K_\bullet^{-k}$ is simply the vector bundle $\mathrm{Hom}(E^k, E^0)$ supported in degree 0. Stability implies

$$s(E^0) \quad < \quad s(E) \quad < \quad s(E^k).$$

If follows that

$$\dim \mathrm{H}^1(K_\bullet)^{-k} \geq -\chi(K_\bullet^{-k}) = -\chi(\mathrm{Hom}(E^k, E^0)) > 0.$$

This completes the proof of Lemma 4.9. □

5. Connectedness of the locus of $\mathrm{U}(p, p)$-binary Hodge bundles

In this section, we show that $\mathcal{B}_{(d_P,d_Q)}$ is irreducible for fixed d_P and d_Q, thus, proving Theorem 1.1.

Proposition 5.1. *If $\tau > 2(p-1)(g-1)$, then $\mathcal{B}_{(d_P,d_Q)}$ is irreducible.*

Recall that the assumption of Lemma 3.4 implies $p = q$ (See Remark 1.2). Key to the proof of Proposition 5.1 is the alternative description of $\mathcal{B}_{(d_P,d_Q)}$ provided by Lemma 3.4. Suppose $(V_P \oplus V_Q, \Phi)$ is a binary Hodge bundle. Then we obtain the length $p(2g-2) + d_Q - d_P$ quotient sheaf $(V_Q, f : V_Q \to V_Q/(\Phi_1(V_P) \otimes \Omega^{-1}))$. Conversely, for each length $p(2g-2) + d_Q - d_P$ quotient sheaf $(E, f : E \to F)$, we obtain a binary Hodge bundle $(\ker(f) \otimes \Omega \oplus E, \Phi_1)$, where

$$\Phi_1 : \ker(f) \otimes \Omega \longrightarrow E \otimes \Omega$$

is the natural inclusion. In this way, Lemma 3.4 provides the following alternative description of the moduli space $\mathcal{B}_{(d_P,d_Q)}$ of binary Hodge bundles: $\mathcal{B}_{(d_P,d_Q)}$ *parameterizes a family of equivalence classes of pairs $(E, f : E \to F)$ of a rank p vector bundle E of degree d_Q and a length $p(2g-2)+d_Q-d_P$ quotient sheaf $f : E \to F$ on X.* The irreducibility of $\mathcal{B}_{(d_P,d_Q)}$ is an easy consequence of this description.

Proof. (of Proposition 5.1) We construct an auxiliary irreducible variety Q_2 and a Zariski open subset $Q_2^{ss} \subset Q_2$. The subset Q_2^{ss} maps surjectively onto $\mathcal{B}_{(d_P,d_Q)}$. The scheme Q_2 is a relative Quot scheme over a subset R of a Quot scheme Q_1 of vector bundles. We recall the construction of R following Seshadri [13].

The space $\mathcal{B}_{(d_P,d_Q)}$ is a subset of the moduli space of Higgs pairs. The family of Higgs bundles parameterized by $\mathbb{C}\mathcal{M}$ is bounded [15]. Hence, there exists an ample line bundle L on X with the following property: *Every Hodge bundle in the family of Hodge bundles parameterized by $\mathcal{B}_{(d_P,d_Q)}$*

admits a representative $(V_P \oplus V_Q, \Phi_1)$ *with vanishing first cohomology* $\mathrm{H}^1(X, V_Q \otimes L) = 0$. Let $H(m) := \chi(V_Q \otimes L^m)$ be the Hilbert polynomial of rank p vector bundles of degree d_Q. Set $a = H(1)$ and let $Q_1 := Quot^H_{\oplus_{i=1}^a L^{-1}/X/\mathbb{C}}$ be the Grothendieck scheme parameterizing the quotient sheaves of $\oplus_{i=1}^a L^{-1}$ with Hilbert polynomial H [9]. The scheme Q_1 contains an irreducible and smooth quasi-projective variety R defined by

$$R = \{W \in Q_1 : W \text{ is locally free and } \mathrm{H}^1(W) = 0\}$$

([13] Chapter III Proposition 23). By our choice of L, every Higgs pair in $\mathcal{B}_{(d_P,d_Q)}$ is represented by a pair $(V_Q \oplus V_P, \Phi)$, such that V_Q is realized in R as a quotient of $\oplus_{i=1}^a L^{-1}$.

Let

$$E \longrightarrow X \times R$$

be the universal quotient bundle of $\oplus_{i=1}^a L^{-1}$. Then there exists a relative Quot scheme

$$Q_2 \ := \ Quot^{-d_P+d_Q+2p(g-1)}_{E/X\times R/R}$$

parameterizing quotient sheaves of E supported as length $-d_P + d_Q + 2p(g-1)$ subschemes of a fiber of $X \times R \to R$ [9]. By construction and Lemma 3.4, Q_2 parameterizes a family of Higgs bundles that contains representatives of all classes in $\mathcal{B}_{(d_P,d_Q)}$.

The morphism $Q_2 \to R$ factors through a surjective morphism

$$h \ : \ Q_2 \ \to \ R \times X^{(-d_P+d_Q+2p(g-1))},$$

where $X^{(m)}$ stands for the m-symmetric product of X. A quotient sheaf F is sent to $\sum_{x\in X} \ell_x \cdot x$, where ℓ_x is the length of the stalk of F at the point x. Each fiber of h is isomorphic to the product of infinitesimal Quot schemes $Q(\ell, \mathcal{O}^p_{(x)})$ of length ℓ quotients of the stalk at x of the trivial rank p vector bundle. Consider the Zariski open subscheme Q_2^{free} of Q_2 parameterizing pairs of quotient sheaves $(\oplus_{i=1}^a L^{-1} \to W, W \to F)$, where F is supported on a subscheme $D \subset X$ as a free $\mathcal{O}_D$-module of rank 1. The restriction of h to Q_2^{free} is a smooth morphism. The scheme Q_2^{free} is irreducible because R, the symmetric product of X, and each fiber of h are irreducible. Q_2^{free} is dense in Q_2, because each fiber of h in Q_2^{free} is dense in the fiber in Q_2 (Lemma 5.2).

Finally, the semi-stable condition is open. Hence, the subscheme Q_2^{ss} of Q_2, parameterizing the semi-stable Higgs bundles, is open and, consequently, irreducible. Proposition 5.1 follows from the fact that Q_2^{ss} admits a surjective morphism onto $\mathcal{B}_{(d_P,d_Q)}$. □

Lemma 5.2. *The Quot scheme $Q(\ell, \mathcal{O}^p_{(x)})$, of length ℓ quotients of the stalk at x of the trivial rank p vector bundle, is irreducible.*

Proof. The Lemma must be well known. We could not find a reference, so we include a short proof. Let m be the maximal ideal of x and A the ring $\mathcal{O}_{(x)}/m^\ell$. Any quotient sheaf in $Q(\ell, \mathcal{O}^p_{(x)})$ is also a quotient of the free A-module of rank p. It is a direct sum of at most p cyclic A-modules (Nakayama's Lemma and the classification of modules over the discrete valuation ring $\mathcal{O}_{(x)}$). Thus, the isomorphism class of a quotient sheaf is determined by a partition of ℓ as a sum of p non-negative integers.

Let $G := GL(p, A)$ be the group of automorphisms of the free A-module of rank p. The group G acts on $Q(\ell, \mathcal{O}^p_{(x)})$. Let $Q(\ell, \mathcal{O}^p_{(x)})^{(\ell)}$ be the orbit of quotient sheaves that are free A-modules of rank 1. It suffices to prove that the orbit $Q(\ell, \mathcal{O}^p_{(x)})^{(\ell)}$ is dense in $Q(\ell, \mathcal{O}^p_{(x)})$.

The proof is by induction on p. If $p = 1$, then $Q(\ell, \mathcal{O}^1_{(x)}) = Q(\ell, \mathcal{O}^1_{(x)})^{(\ell)}$ is a point. Let $(\ell_1, \ell_2, \dots, \ell_p)$, $\ell_i \geq 0$, be a partition of ℓ. We show that the orbit $Q(\ell, \mathcal{O}^p_{(x)})^{(\ell_1, \ell_2, \dots, \ell_p)}$ is in the closure of $Q(\ell, \mathcal{O}^p_{(x)})^{(\ell)}$. If some of the ℓ_i are zero, the proof reduces to the case of a smaller p. Assume that none of the ℓ_i vanishes. Let z be a local parameter and consider the linear combination $\eta(t) := \psi + t\varphi$, $t \in \mathbb{C}$, of the two $p \times p$ matrices

$$\psi = \begin{pmatrix} 0 & & \cdots & 0 & z^{\ell_p} \\ z^{\ell_1} & 0 & \cdots & & 0 \\ 0 & z^{\ell_2} & 0 \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & z^{\ell_{p-1}} & 0 \end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & \\ & 0 & & \vdots & \vdots \\ \vdots & \vdots & & 1 & 0 \\ & 0 & \cdots 0 & 1 & 0 \\ 0 & 0 & \cdots 0 & 0 & z^\ell \end{pmatrix}.$$

When $t = 0$, $\eta(0)$ is equal to ψ and its cokernel is a quotient sheaf in the orbit $Q(\ell, \mathcal{O}^p_{(x)})^{(\ell_1, \ell_2, \dots, \ell_p)}$. If $t \neq 0$, then row reduction leads to the expression of the column $\eta(t)_p$ as a linear combination of the first $p - 1$ columns (mod z^ℓ):

$$\begin{aligned} \eta(t)_p \;=\; & \frac{z^{\ell_p}}{t}\eta(t)_1 - \sum_{k=2}^{p-1} \left[\left(\frac{-1}{t} \right)^k z^{(\ell_p + \sum_{i=1}^{k-1} \ell_i)} \right] \eta(t)_k \\ & + \left[t + \left(\frac{-1}{t} \right)^{p-1} \right] \cdot z^\ell \cdot e_p, \end{aligned}$$

where $\{e_1, \dots, e_p\}$ is the standard basis of $\mathcal{O}^p_{(x)}$. Consequently, if $t \neq 0$ and $t^p \neq (-1)^p$, the cokernel of $\eta(t)$ is in $Q(\ell, \mathcal{O}^p_{(x)})^{(\ell)}$. □

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