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The Moduli of Flat $\mathrm{U}(p, 1)$ Structures on Riemann Surfaces

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Abstract. For X a smooth projective curve over $\mathbb{C}$ of genus $g > 1$, $\mathrm{Hom}^+(\pi_1(X), \mathrm{U}(p, 1))/\mathrm{U}(p, 1)$ is the moduli space of flat semi-simple $\mathrm{U}(p, 1)$-connections on X. There is an integer invariant, τ, the Toledo invariant associated with each element in $\mathrm{Hom}^+(\pi_1(X), \mathrm{U}(p, 1))/\mathrm{U}(p, 1)$. This paper shows that $\mathrm{Hom}^+(\pi_1(X), \mathrm{U}(p, 1))/\mathrm{U}(p, 1)$ has one connected component corresponding to each $\tau \in 2\mathbb{Z}$ with $-2(g-1) \leqslant \tau \leqslant 2(g-1)$. Therefore the total number of connected components is $2(g-1)+1$.

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1. Introduction and Results

Let X be a smooth projective curve over $\mathbb{C}$ with genus $g > 1$. The deformation space

$$\mathbb{C}\mathcal{M}_B = \mathrm{Hom}^+(\pi_1(X), \mathrm{GL}(n, \mathbb{C}))/\mathrm{GL}(n, \mathbb{C})$$

is the space of equivalence classes of semi-simple $\mathrm{GL}(n, \mathbb{C})$-representations of the fundamental group $\pi_1(X)$. This is the $\mathrm{GL}(n, \mathbb{C})$-Betti moduli space on X. A theorem of Corlette, Donaldson, Hitchin and Simpson relates $\mathbb{C}\mathcal{M}_B$ to two other moduli spaces: the $\mathrm{GL}(n, \mathbb{C})$-de Rham and the $\mathrm{GL}(n, \mathbb{C})$-Dolbeault moduli spaces, respectively [3, 4, 10, 18]. The Dolbeault moduli space consists of holomorphic objects (Higgs bundles) over X; therefore, the classical results of analytic and algebraic geometry can be applied to the study of the Dolbeault moduli space.

Let $n = p + 1$. Since $\mathrm{U}(p, 1) \subset \mathrm{GL}(n, \mathbb{C})$, $\mathbb{C}\mathcal{M}_B$ contains the space

$$\mathcal{M}_B = \mathrm{Hom}^+(\pi_1(X), \mathrm{U}(p, 1))/\mathrm{U}(p, 1).$$

The space $\mathcal{M}_B$ will be referred to as the $\mathrm{U}(p, 1)$-Betti moduli space.

The Betti moduli spaces are of great interest in geometric topology and uniformization. When $p = q = 1$, Goldman analyzed $\mathcal{M}_B$ and determined the number of its connected components [5]. Hitchin subsequently considered this moduli space from the Higgs bundle perspective and determined its topology [10]. The case of $p = 2, q = 1$ was treated in [26]. Other related results have been obtained in [6, 24, 25]. In this paper, we treat the general case of $q=1$ and determine its number of connected components.

Each element in $\mathcal{M}_B$ is associated with a Toledo invariant $\tau \in 2\mathbb{Z}$ which is bounded as [4, 22, 23, 26]

$$-2(g-1) \leqslant \tau \leqslant 2(g-1).$$

The main result presented here is the following:

THEOREM 1.1. $\mathrm{Hom}^+(\pi_1(X), \mathrm{U}(p,1))/\mathrm{U}(p,1)$ *has one connected component for each* $\tau \in 2\mathbb{Z}$ *with* $-2(g-1) \leqslant \tau \leqslant 2(g-1)$. *Therefore the total number of connected components is* $2(g-1)+1$.

The cases of $p=1,2$ was treated in [5, 10, 26]. More recently, Gothen computed the Poincaré polynomial for the smooth components in the case of $p=2$ [7]. The case of $\mathrm{PU}(p,p)$-representations for large τ was treated in [12].

After the completion of this paper, S. Bradlow, O. Garcia-Prada and P. Gothen announced that all moduli spaces of flat $\mathrm{PU}(p,q)$-structures with fixed Chern class and Toledo invariant, are connected [2].

2. Backgrounds and Preliminaries

The coarse moduli space $M_{r,d}$ of semi-stable vector bundles on X of rank r and degree d exists and has dimension $r^2(g-1)+1$ [16].

2.1. THE $\mathrm{U}(p,1)$-HIGGS BUNDLES

Each element $\rho \in \mathrm{Hom}(\pi_1(X), \mathrm{GL}(n,\mathbb{C}))$ acts on $\mathbb{C}^n$ via the standard representation of $\mathrm{GL}(n,\mathbb{C})$. The representation ρ is called reducible (irreducible) if its action on $\mathbb{C}^n$ is reducible (irreducible). A representation ρ is called semi-simple if it is a direct sum of irreducible representations.

DEFINITION 2.1.

$$\mathbb{C}\mathcal{M}_B = \{\sigma \in \mathrm{Hom}(\pi_1(X), \mathrm{GL}(n,\mathbb{C})) : \sigma \text{ is semi-simple}\}/\mathrm{GL}(n,\mathbb{C}).$$

$$\mathcal{M}_B = \{\sigma \in \mathrm{Hom}(\pi_1(X), \mathrm{U}(p,1)) : \sigma \text{ is semi-simple}\}/\mathrm{U}(p,1).$$

Let V be a rank-n holomorphic bundle over X with $\deg(V)=0$. Denote by Ω the canonical bundle on X. A Higgs bundle is a pair (V,Φ), where $\Phi \in \mathrm{H}^0(X, \mathrm{End}(V)\otimes\Omega)$. Such a Φ is called a Higgs field. Define the slope of V to be

$$s(V) = \deg(V)/\mathrm{rank}(V).$$

For a fixed Φ, a holomorphic sub-bundle $W \subset V$ is said to be Φ-invariant if $\Phi(W) \subset W \otimes \Omega$. A Higgs bundle (V,Φ) is stable (semi-stable) if $W \subset V$ being Φ-invariant implies $s(W) < (\leqslant) s(V)$. A Higgs bundle is called poly-stable if it is a direct sum of stable Higgs bundles of the same slope [10, 19].

The Dolbeault moduli space $\mathbb{C}\mathcal{M}$ is the moduli space of the S-equivalence classes of semi-stable Higgs bundles. The complex points of this moduli space are naturally the isomorphism classes of the poly-stable Higgs bundles [10, 11, 15, 19]. A Higgs bundle is called reducible if it is poly-stable (semi-stable) but not stable.

Now we summarize the relation between the moduli spaces $\mathbb{C}\mathcal{M}_B$ and $\mathbb{C}\mathcal{M}$.

DEFINITION 2.2. Let $\mathcal{M}$ be the subset of $\mathbb{C}\mathcal{M}$ consisting of Higgs bundles (V, Φ) satisfying the following two conditions:

(1) V is a direct sum: $V = V_P \oplus V_Q$, where V_P, V_Q are of ranks p, 1, respectively.
(2) The Higgs field decomposes into two maps:

$$\Phi_1 \colon V_P \longrightarrow V_Q \otimes \Omega,$$

$$\Phi_2 \colon V_Q \longrightarrow V_P \otimes \Omega.$$

Hence each $(V, \Phi) = (V_P \oplus V_Q, \Phi) \in \mathcal{M}$ is associated with an invariant

$$d = \deg(V_P) = -\deg(V_Q).$$

The Toledo invariant is defined to be $\tau = 2d$. The subset of $\mathcal{M}$ consisting of classes with a fixed Toledo invariant τ is denoted by $\mathcal{M}_\tau$.

THEOREM 2.3.

(1) *The moduli spaces* $\mathbb{C}\mathcal{M}_B$ *and* $\mathbb{C}\mathcal{M}$ *are homeomorphic.*
(2) *The reducible representations in* $\mathbb{C}\mathcal{M}_B$ *correspond to the poly(semi)-stable, but not stable, points.*
(3) *The subspace* $\mathcal{M}_B$ *is homeomorphic to* $\mathcal{M}$.
(4) $\tau \in 2\mathbb{Z}$ *and* $-2(g-1) \leqslant \tau \leqslant 2(g-1)$.
(5) *The space* $\mathcal{M}_\tau$ *is homeomorphic to* $\mathcal{M}_{-\tau}$.

Proof. See [3, 10, 18, 19, 22, 23, 26]. □

By Theorem 2.3 (4) (5), we may assume that

$$0 \leqslant \tau \leqslant 2(g-1)$$

3. The $\mathbb{C}^*$-Action and the Hodge Bundles

If $(V, \Phi) \in \mathbb{C}\mathcal{M}$, then $(V, t\Phi) \in \mathbb{C}\mathcal{M}$ for all $t \in \mathbb{C}^*$. This defines an action [10, 11, 19] $\mathbb{C}^* \times \mathbb{C}\mathcal{M} \longmapsto \mathbb{C}\mathcal{M}$. By Definition 2.2, we have

PROPOSITION 3.1. *The $\mathbb{C}^*$-action preserves $\mathcal{M}$.*

A Hodge bundle on X is a direct sum of bundles [19]

$$V = \bigoplus_{s,t} V^{s,t}$$

together with maps (Higgs field)

$$\Phi_{s,t} \colon V^{s,t} \longrightarrow V^{s-1,t+1} \otimes \Omega.$$

Definition 2.2 implies that

PROPOSITION 3.2. *Suppose $(V_P \oplus V_Q, (\Phi_1, \Phi_2)) \in \mathcal{M}$. Then $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ is a Hodge bundle if and only if $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ is either binary or ternary in the following sense:*

(1) *Binary: $\Phi_2 \equiv 0$.*
(2) *Ternary: $V_P = V_1 \oplus V_2$ and the Higgs field consists of two maps:*

$$\Phi_1 \colon V_2 \longrightarrow V_Q \otimes \Omega,$$

$$\Phi_2 \colon V_Q \longrightarrow V_1 \otimes \Omega.$$

PROPOSITION 3.3. *A Higgs bundle $(V, \Phi) \in \mathbb{C}\mathcal{M}$ is a Hodge bundle if and only if $(V, \Phi) \cong (V, t\Phi)$ for all $t \in \mathbb{C}^*$.*

Proof. See [19–21]. □

LEMMA 3.4. *The $\lim_{t\to 0}(V, t\Phi)$ exists in $\mathcal{M}$ for any (V, Φ) in $\mathcal{M}$. In other words, the $\mathbb{C}^*$-action always extends to a $\mathbb{C}$-action.*

Proof. The $\lim_{t\to 0}(V, t\Phi)$ exists in $\mathbb{C}\mathcal{M}$ for (V, Φ) in $\mathbb{C}\mathcal{M}$ [21]. Since $\mathrm{U}(p, 1)$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, $\mathcal{M}_B$ is a closed subset of $\mathbb{C}\mathcal{M}_B$ and the embedding is proper [26]. The lemma then follows from Theorem 2.3 and Proposition 3.1. □

From Proposition 3.3 and Lemma 3.4 and the facts that the $\mathbb{C}^*$-action preserves $\mathcal{M}$ and $\mathcal{M}$ is closed in $\mathbb{C}\mathcal{M}$, we have

COROLLARY 3.5. *Every Higgs bundle in $\mathcal{M}$ can be deformed to a Hodge bundle in $\mathcal{M}$.*

LEMMA 3.6. *Let $E = (V, \Phi)$ be a stable Hodge bundle in $\mathcal{M}$ that is not binary. Then there is a stable Higgs bundle $F = (V', \Phi') \in \mathcal{M}$ not isomorphic to E such that $\lim_{t\to\infty}(V', t\Phi') = E$.*

Proof. This is essentially Lemma 11.9 in [21]. There is one additional and crucial observation to be made. That is the Higgs bundle F in the proof is actually in $\mathcal{M}$. This is immediate from the construction of F. □

PROPOSITION 3.7. *Every Higgs bundle in $\mathcal{M}$ can be deformed to a binary Hodge bundle.*

Proof. The proof parallels the proof of Corollary 11.10 in [21]. The only difference is that for the subset $\mathcal{M}_\tau$, the lowest stratum is the space of binary Hodge bundles with Toledo invariant τ, instead of $M_{p+1,0}$. Suppose $(V,\Phi)\in\mathcal{M}_\tau$. Then it is of the form $(W_1,0)\oplus(W_2,\Phi_2)$ where (W_2,Φ_2) is stable. By Corollary 3.5, we may assume (W_2,Φ_2) is not binary. Then Lemma 3.6 implies that (W_2,Φ_2) is not in the lowest possible stratum, hence, can be deformed to a fixed point set (with respect to the $\mathbb{C}^*$-action) of the lowest stratum which consists of binary Higgs bundles (See the section titled 'Actions of $\mathbb{C}^*$' in [21] for further discussions of these strata). □

Remark 3.8. Proposition 3.7 is also true when $\mathrm{rank}(V_q)>1$. These Higgs bundles correspond to U(p,q)-representations.

DEFINITION 3.9. Let B_τ be the space of all poly-stable (or S-equivalence classes of semi-stable [15]) binary Hodge bundles $(V_P\oplus V_Q,(\Phi_1,0))$ with $\deg(V_P)=d=-\deg(V_Q)$ and $\tau=2d$.

The rest of the paper is devoted to showing that B_τ is connected.

4. The Deformation of Binary Hodge Bundles

The fact that $\mathbb{C}\mathcal{M}$ is a moduli space implies that if Z is a family of stable (poly-stable or S-equivalence classes of semi-stable) Higgs bundles, then there is a natural morphism $Z\longrightarrow\mathbb{C}\mathcal{M}$ which takes every point $z\in Z$ to the point of $\mathbb{C}\mathcal{M}$ that corresponds to the Higgs bundle in the family over z [13, 14, 15]. The space $\mathcal{M}$ is a subvariety of $\mathbb{C}\mathcal{M}$; hence, to show that two stable (poly-stable or S-equivalence classes of semi-stable) Higgs bundles (V_1,Φ_1) and (V_2,Φ_2) belong to the same component of $\mathcal{M}$, it suffices to exhibit a connected family Z of stable (poly-stable or S-equivalence classes of semi-stable) Higgs bundles containing both (V_1,Φ_1) and (V_2,Φ_2). The strategy is as follow: First construct certain Quot schemes each of which contains a connected open subscheme that parameterizes a family of Higgs bundles in $\mathcal{M}$; moreover, the union of these families contains all Higgs bundles parameterized by $\mathcal{M}$. Then construct connected schemes (scheme Y below) that connect these subschemes of the Quot schemes.

4.1. THE GROTHENDIECK QUOT SCHEME

Denote by H_{r,d_1} the set of all vector bundles of rank r and degree $d_1\leqslant 0$ with the property that if $W\in H_{r,d_1}$ and $U\subset W$, then $\deg(U)\leqslant 0$.

PROPOSITION 4.1. *Suppose $W\in H_{r,d_1}$. Then for any line bundle L with* $\deg(L)>2g-1-d_1$,

(1) $\mathrm{H}^1(W\otimes L)=0$,
(2) *$W\otimes L$ is generated by global sections.*

Proof. The proof is an adaptation of the proof of Lemma 20 in Chapter I of [16]. By Serre duality, $\mathrm{H}^1(W \otimes L) \cong \mathrm{H}^0(\Omega \otimes W^* \otimes L^*)$. Hence if $\mathrm{H}^1(W \otimes L) \neq 0$, then $\Omega \otimes W^* \otimes L^*$ contains a line bundle N of degree greater than or equal to 0:

$$0 \longrightarrow N \longrightarrow \Omega \otimes W^* \otimes L^* \longrightarrow (\Omega \otimes W^* \otimes L^*)/N \longrightarrow 0.$$

Dualizing and tensoring with $\Omega \otimes L^*$, we obtain an exact sequence

$$0 \longrightarrow U \longrightarrow W \longrightarrow L' \longrightarrow 0,$$

with $\deg(U) \geq d_1 - (2g-2) + \deg(L)$. Since $W \in H_{r,d_1}$ and $U \subset W$, $\deg(U) \leqslant 0$. This implies $\deg(L) \leqslant 2g-2-d_1$ which is a contradiction. This shows that (1) is true for any $L > 2g-2-d_1$.

The proof of (2) essentially reduces to showing that

$$\mathrm{H}^1(W \otimes L \otimes L_x^{-1}) = 0$$

(see the proof of Lemma 20 in Chapter I of [16]), where L_x is the ideal sheaf at a point $x \in X$. Since $\deg(L_x) = 1$, $\deg(L \otimes L_x^{-1}) > 2g-2-d_1$. Since (1) is true for any $L > 2g-2-d_1$, (2) follows. □

Let $D = 2gr + (1-r)d_1$ and $a = D + r(1-g) = r(g+1) + (1-r)d_1$. For the pair (a, r), we construct the Grothendieck scheme Q parameterizing the quotient sheaves of $\mathcal{O}^a$ with Hilbert polynomial $H(m) = a + rm$ [8]. The scheme Q contains the sub-scheme R defined by

$$R = \{W \in Q : W \text{ is locally free and } \mathrm{H}^1(W) = 0\}.$$

The sub-scheme R is smooth and connected (The proof is the same as that of Proposition 23 in Chapter I of [16]).

Suppose L is a line bundle of degree $-2g + d_1$. If $W \in R$, then $\deg(W \otimes L) = d_1$. Hence R also parameterizes a family of vector bundles of degree d_1 and rank r. By Proposition 4.1, R contains all the bundles in H_{r,d_1}. We shall denote the scheme R so constructed as R_{r,d_1}.

4.2. THE CANONICAL FACTORIZATION

Let $(V_P \oplus V_Q, \Phi) \in B_\tau$ be a binary Higgs bundle with $\Phi \not\equiv 0$. There exist bundles V_1, V_2 and W_1 such that the following diagram (the canonical factorization [17])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{f_1} & V_P & \xrightarrow{f_2} & V_2 & \longrightarrow & 0 \\ & & & & \Phi\downarrow & & \varphi\downarrow & & \\ & & & & V_Q \otimes \Omega & = & W_1 & & \end{array}$$

commutes, with $\mathrm{rank}(V_2) = \mathrm{rank}(W_1)$ and φ has full rank at a generic point of X. Since $(V_P \oplus V_Q, \Phi)$ is semi-stable, $d_1/(p-1) \leqslant 0$. Since $\varphi \not\equiv 0$, $d_2 \leqslant (2g-2) - d$ (Recall that $d = \deg(V_P)$). Hence $d_1 \geqslant 2d - (2g-2)$. To summarize

$$2d-(2g-2)\leqslant d_1\leqslant 0,$$
$$d\leqslant d_2\leqslant -d+(2g-2).$$

Denote by $B_\tau(d_2)$ the subspace of B_τ such that $(V,\Phi)\in B_\tau(d_2)$ implies $\deg(V_2)=d_2$ in the above canonical factorization.

PROPOSITION 4.2 *The space $B_\tau(d_2)$ is connected.*

Proof. Denote by J^l the Jacobi variety identified with the set of holomorphic line bundles of degree l on X. For each $V_2\in J^{d_2}$, the variety $\mathbb{C}^*\times X^{-d+2(g-1)-d_2}$ parameterizes a family of pairs that contains all pairs (V_Q,φ) such that $V_Q\in J^{-d}$ and

$$0\not\equiv\varphi\in \mathrm{H}^0(X,V_2^{-1}\otimes V_Q\otimes\Omega).$$

Note the moduli of all such pairs is simply $\mathbb{C}^*\times \mathrm{Sym}^{-d+2(g-1)-d_2}X$, where Sym^tX is the t-th symmetric product of X, i.e. the moduli is $\mathbb{C}^*\times X^{-d+2(g-1)-d_2}$ quotiented by the symmetry group on $-d+2(g-1)-d_2$ letters. Hence, the variety

$$S=J^d\times(\mathbb{C}^*\times X^{-d+2(g-1)-d_2})$$

parameterizes a family of triples that contains all the triples (V_2,V_Q,φ) such that

$$V_2\overset{\varphi}{\longmapsto}V_Q\otimes\Omega,$$

with $V_2\in J^{d_2}$, $V_Q\in J^{-d}$ and $\varphi\not\equiv 0$. The variety S is smooth.

Suppose $d_2=0$. Then $d_1=0$ and every Higgs bundle in $B_\tau(0)$ is reducible and $d=0$. It is immediate that every Higgs bundle in $B_0(0)$ must be one of the following two forms:

(1) $(V_P\oplus V_Q,0)$, where $V_P\in M_{p,0}$, $V_Q\in J^0$,
(2) $(V_1,0)\oplus(V_2\oplus V_Q,\Phi)$, where $V_1\in M_{p-1,0}$, $V_2,V_Q\in J^0$.

Type (2) can be deformed to type (1) by simply deforming the Higgs field Φ to zero. Since $M_{p,0}\times J^0$ is connected, $B_0(0)$ is connected.

LEMMA 4.3. *Suppose $d_2>0$. Then the dimension of the space* $\mathrm{Ext}^1(V_2,V_1)$ *is* $(p-1)(g-1)+(p-1)d_2-d_1$.

Proof. The subspace V_1 is Φ-invariant. By semi-stability, $V'\subset V_1$ implies

$$s(V')\leqslant 0<d_2=\deg(V_2).$$

Hence $\mathrm{H}^0(\mathrm{Hom}(V_2,V_1))=0$. The lemma then follows from the fact that $\mathrm{Ext}^1(V_2,V_1)\cong \mathrm{H}^1(\mathrm{Hom}(V_2,V_1))$ and from Riemann–Roch. □

For $d_2>0$, construct the universal bundle [1, 16]

$$U\longrightarrow X\times R_{p-1,d_1}\times S$$

such that

$$U\big|_{(X,V_1,V_2,V_Q,\varphi)}=V_2^{-1}\otimes V_1.$$

Let π be the projection

$$\pi\colon X \times R_{p-1,d_1} \times S \longrightarrow R_{p-1,d_1} \times S.$$

Applying the right derived functor R^1 to π gives the sheaf $\mathcal{F} = R^1\pi_*(U)$ [9] such that

$$\mathcal{F}\big|_{(V_1,V_2,V_Q,\varphi)} = \mathrm{H}^1(X, V_2^{-1} \otimes V_1).$$

By Lemma 4.3 and Grauert's theorem [9], $\mathcal{F}$ is locally free, hence, is associated with a vector bundle

$$F \longmapsto R_{p-1,d_1} \times S$$

of rank $(p-1)(g-1) + (p-1)d_2 - d_1$. Since R_{p-1,d_1} is smooth and connected, the total space F is smooth, connected and parameterizes a family of Higgs bundles that fit into the canonical decomposition with fixed d_2. By construction, the scheme F parameterizes a family of Higgs bundles that contains every member in the parameter space $B_\tau(d_2)$. Moreover if a Higgs bundle in F is semi-stable, it must belong to $B_\tau(d_2)$. Since the semi-stability condition is open [20], the subset of F parameterizing the semi-stable Higgs bundles, if not empty, is open and dense in F, hence, connected. This implies $B_\tau(d_2)$ is connected. □

4.3. DEFORMATION BETWEEN THE $B_\tau(d_2)$'S

Fix a set of distinct points

$$A = \{x_1, \dots, x_{d_2}, y_1, \dots, y_{d_2-1}, z_1, \dots, z_{d_2-d-1}\} \subset X$$

and let $Y = X\backslash A$. Fix $y \in Y$. For $t \in Y$, consider the following divisors on X:

$$D_2 = \sum_{i=1}^{d_2} x_i, \qquad C(t) = D_2 - t - \sum_{i=1}^{d_2-1} y_i, \qquad C = D_2 - y - \sum_{i=1}^{d_2-d-1} z_i.$$

The set Y parameterizes a family of Higgs bundles as follows. Let

$$V_P(t) = \mathcal{O}(C) \bigoplus_{i=1}^{p-1} \mathcal{O}(C_i(t)),$$

where $C_i(t) = C(t)$ for all i and denote the projection maps to the $\mathcal{O}(C)$ and $\mathcal{O}(C_i(t))$ factors by p and $p_i(t)$, respectively. The divisors $D_2 - C(t)$ and $D_2 - C$ define maps $h_i(t)\colon \mathcal{O}(C_i(t)) \longrightarrow \mathcal{O}(D_2)$ and $h\colon \mathcal{O}(C) \longrightarrow O(D_2)$, respectively. These maps induce a map

$$G_t\colon V_P(t) \longrightarrow O(D_2), \qquad G_t = h + \sum_{i=1}^{p-1} h_i(t).$$

Let $V_2 = \mathcal{O}(D_2)$. Since $d_2 \leqslant (2g-2) - d$, there exists $V_Q \in J^{-d}$ and $0 \not\equiv \varphi \in \mathrm{H}^0(V_2^{-1} \otimes V_Q \otimes \Omega)$. Let $\Phi(t) = \varphi \circ G_t$. Then $(V_P(t) \oplus V_Q, \Phi(t))$ is a family of Higgs bundles parameterized by Y. Let p_P, p_Q be the projections onto the $V_P(t), V_Q$ factors.

LEMMA 4.4. *If* $U \subset V_P(t)$, *then* $\deg(U) \leqslant d$.

Proof. This is an inductive argument.

Case 1: $p(U) \neq 0$. Consider the sequence

$$0 \longrightarrow U' \longrightarrow U \longrightarrow p(U) \longrightarrow 0.$$

Then $\deg(p(U)) \leqslant d$ and $\deg(U) \leqslant \deg(U') + d$. Now we begin with the smallest i with $p_i(t)(U') \neq 0$ and construct

$$0 \longrightarrow U_i \longrightarrow U' \longrightarrow p_i(U') \longrightarrow 0.$$

Again $\deg(p_i(U')) \leqslant \deg(\mathcal{O}(C_i(t))) = 0$ and $\deg(U) \leqslant \deg(U_i) + d$. Now we let $j > i$ be the smallest integer with $p_j(U_i) \neq 0$ and construct the new sequence and obtaining U_j with $\deg(U) \leqslant \deg(U_j) + d$. Note $\operatorname{rank}(U) > \operatorname{rank}(U') > \operatorname{rank}(U_i)$, so eventually the process ends and since $\deg(O(C_i(t))) = 0$ for all i, we have $\deg(U) \leqslant d$.

Case 2: $p(U) = 0$. Here we simply begin with the smallest i with $p_i(t)(U') \neq 0$ as in Case 1. The rest is the same and we conclude that $\deg(U) \leqslant 0 \leqslant d$. □

LEMMA 4.5. *Suppose* $L \subset V_P(t)$ *is a line bundle with* $\deg(L) > 0$, *then* $L = \mathcal{O}(C)$.

Proof. Suppose $p_i(t)(L) \not\equiv 0$ for some i, then $\deg(L) \leqslant \deg(C_i(t)) = 0$, a contradiction. □

PROPOSITION 4.6. *The Higgs bundle* $(V_P(t) \oplus V_Q, \Phi(t))$ *is in* $B_\tau(d_2 - 1)$ *if* $t = y$ *and in* $B_\tau(d_2)$ *if* $t \neq y$.

Proof. From the definition, it is sufficient to check that $(V_P(t) \oplus V_Q, \Phi(t))$ is semi-stable for all $t \in Y$, i.e. the Higgs bundles parameterized by Y belong to the family parameterized by $\mathcal{M}$.

Suppose $W \in V_P(t) \oplus V_Q$ is $\Phi(t)$ invariant. There are two cases.

Case 1: $p_Q(W) \neq 0$. Then there is an exact sequence

$$0 \longrightarrow U \longrightarrow W \longrightarrow p_Q(W) \longrightarrow 0.$$

Since $U \subset V_P$, by Lemma 4.4, $\deg(U) \leqslant d$. Since $\deg(p_Q(W)) \leqslant \deg(V_Q) = -d$, $\deg(W) \leqslant d - d = 0$.

Case 2: $p_Q(W) = 0$. Since W is $\Phi(t)$-invariant, $W \subset \ker(G_t)$. It is immediate from the definition that, $\mathcal{O}(C) \not\subset \ker(G_t)$. Now we begin the construction similar to that in the proof of Lemma 4.4. Begin with the smallest i with $p_i(t)(W) \neq 0$ and construct

$$0 \longrightarrow U_i \longrightarrow W \longrightarrow p_i(W) \longrightarrow 0.$$

As $\deg(p_i(W)) \leqslant \deg(\mathcal{O}(C_i(t))) = 0$, $\deg(W) \leqslant \deg(U_i)$. Continue this process as before. Eventually we reach the exact sequence

$$0 \longrightarrow U_k \longrightarrow U_j \longrightarrow p_k(U_j) \longrightarrow 0,$$

with

$$\deg(W) \leqslant \deg(U_j) = \deg(U_k) + \deg(p_k(U_j)) \leqslant \deg(U_k)$$

and U_k a line bundle. Since $\mathcal{O}(C) \not\subset \ker(G_t)$ and $U_k \subset \ker(G_t)$, by Lemma 4.5, $\deg(U_k) \leqslant 0$. Hence $\deg(W) \leqslant \deg(U_k) \leqslant 0$. Hence we conclude that the Higgs bundle $(V_P(t) \oplus V_Q, \Phi(t))$ is semi-stable. □

Proposition 3.7 states that every Higgs bundle in $\mathcal{M}$ may be deformed to a Higgs bundle in B_τ which is the union of the $B_\tau(d_2)$'s. Proposition 4.2 asserts that each $B_\tau(d_2)$ is connected. Theorem 1.1 then follows from Propositon 4.6.

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