1. As follow:

<table>
<thead>
<tr>
<th>Column ⇒ Row</th>
<th>unif.</th>
<th>$L^1$</th>
<th>measure</th>
<th>a.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>unif.</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>$L^1$</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>measure</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>a.e.</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

(1). $f_n$ uniform converge to $f$, so $|f_n - f| < \epsilon$. Hence $\int_a^b |f_n - f| \, dx < \epsilon$.

(2). Because $f_n \to f$ uniform, $m\{x : |f(x) - f_n(x)| > \epsilon \} = 0$.

(3). by definition.

(4). $f_n(x) = x^n$

(5). $m\{x : |f(x) - f_n(x)| > \epsilon \}, \int |f(x) - f_n(x)| \, dx \to 0$ as $n \to \infty$.

(6). $f_1(x) = 1$, $f_2(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$, $f_3(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$.

(7). $f_n(x) = x^n$

(8). $f_n(x) = \begin{cases} n^2 & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$

(9). $f_1(x) = 1$, $f_2(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$, $f_3(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$.

(10). $f_n(x) = x^n$

(11). $f_n(x) = \begin{cases} n^2 & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$

(12). By Egoroff’s Theorem

2. As follow:

<table>
<thead>
<tr>
<th>Column ⇒ Row</th>
<th>conti.</th>
<th>B.V.</th>
<th>abs. conti.</th>
<th>int.</th>
</tr>
</thead>
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<tr>
<td>conti.</td>
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<tr>
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<td>F</td>
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</table>
(1). \( f(x) = \begin{cases} \frac{1}{2} x \sin \left( \frac{1}{x} \right) & \text{if } x \in A \\ 0 & \text{if } x = 0 \end{cases} \)

(2). \( f(x) = \begin{cases} \frac{1}{2} x \sin \left( \frac{1}{x} \right) & \text{if } x \in A \\ 0 & \text{if } x = 0 \end{cases} \)

(3). conti. \( \Rightarrow \) int. on any compact set.

(4). \( f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in \left( \frac{1}{2}, 1 \right] \end{cases} \)

(5). \( f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in \left( \frac{1}{2}, 1 \right] \end{cases} \)

(6). \( f(x) \in B.V. \) then \( f(x) = f_1 - f_2 \), where \( f_1, f_2 \) increasing.

And increasing function is Riemann integrable function.

(7). by definition.

(8). by definition.

(9). abs. conti. \( \Rightarrow \) conti. \( \Rightarrow \) int.

(10). Dirichlet function

(11). Dirichlet function

(12). Dirichlet function

3. As follow:

<table>
<thead>
<tr>
<th>Column ( \Rightarrow ) Row</th>
<th>C.F.T.(I)</th>
<th>C.F.T.(II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>conti.</td>
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<td>F</td>
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</tr>
</tbody>
</table>

(1). conti. \( \Rightarrow \) int. and by C.F.T.(I)

(2). Cantor-Lebesgue function

(3). B.V. \( \Rightarrow \) int. and by C.F.T.(I)

(4). Cantor-Lebesgue function

(5). abs. conti. \( \Rightarrow \) int. and by C.F.T.(I)

(6). by C.F.T.(II)

(7). by C.F.T.(I)

(8). Cantor-Lebesgue function

4. State **LDCT**. If \( f_n \in L \) and \( f_n \rightarrow f \) (a.e.) and \( |f_n| \leq g(x) \in L \),

\[
\lim_{n \to \infty} \int f_n(x) \, dx = \int \lim_{n \to \infty} f_n(x) \, dx
\]

\[
\left| \frac{n \cos x}{1 + n^2 x^2} \right| = \left| \cos x \frac{1}{1 + n \frac{x^2}{2}} \right| \leq \left| \frac{1}{1 + n \frac{x^2}{2}} \right| \leq \left| \frac{1}{x^2} \right| \text{ and } \lim_{n \to \infty} \frac{n \cos x}{1 + n^2 x^2} \rightarrow 0.
\]
Then $\frac{1}{x^2}$ is Riemann integrable on $[1, 2] \implies \frac{1}{x^2} \in L[1, 2]$

By LDCT, $\lim_{n \to \infty} \int_{1}^{2} \frac{n \cos x}{1 + n^2 x^2} \, dx = \int_{1}^{2} \lim_{n \to \infty} \frac{n \cos x}{1 + n^2 x^2} \, dx = \int_{1}^{2} 0 \, dx = 0$.

5. (a) False. Cantor’s set is uncountable and of measure zero.

   (b) False.

   All rational number on $[0,1]$ is countable, therefore all rational number on $[0,1]$ represent $\{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots\}$

   Let $\varphi_n(x) = \begin{cases} 1, & \text{if } x = \gamma_1, \gamma_2, \ldots, \gamma_n \\ 0, & \text{otherwise} \end{cases}$, then $\forall \varphi_n(x) \in \mathcal{R}[0, 1]$

   $\varphi_n(x) \to$ Dirichlet function($\text{Dirichlet function } \notin \mathcal{R}[0, 1]$)

   Hence the space of all Riemann integrable functions is not complete in $L^1$-norm.

(c) False. $f(x) = \frac{1}{x}$ is measurable but not integrable.

(d) True.

   $f(x)$ is bounded variation function. Hence, $f'(x)$ is integrable.

   $f(x) = h(x) + g(x)$, where $h(x) = \int_{a}^{x} f'(t) \, dt$ and $g(x) = f(x) - \int_{a}^{x} f'(t) \, dt$

   $h(x)$ is abs. conti. by C.F.T.(I).

   $g'(x) = f'(x) - f'(x) = 0$ by C.F.T.(I).