Probability

1. (10%) Find the expectation and variance of the random variable X if the distribution function of X is given by

\[ P(x) = \begin{cases} 
0, & \text{if } x < 0, \\
1 - \frac{3}{5} e^{-x}, & \text{if } x \geq 0.
\end{cases} \]

2. (15%) Let X be a random variable. Show that if \( \text{Var}(X) = 0 \), then \( P(X = E(X)) = 1 \).

3. Let X be a random variable distributed as negative binomial with p.d.f. (or p.m.f.)

\[ f(x; r, p) = p^r \left( \frac{r + x - 1}{x} \right) (1 - p)^x, \quad x = 0, 1, \ldots, \quad 0 < p < 1, r = 1, 2, \ldots, \]

and let \( g(x) \) be a function with \( -\infty < E(g(X)) < +\infty \) and \( g(-1) = 0 \).

(a) (10%) Show that

\[ \mathbb{E}(X) = \mathbb{E} \left[ \frac{X}{r + X - 1} g(X - 1) \right]. \]

(b) (10%) Use (a) to find the expectation of X.

4. Let \( X_1, X_2, \ldots, X_n \) be independent random variables distributed as \( P(\lambda_1), P(\lambda_2), \ldots, P(\lambda_n) \), respectively. Let \( T = \sum_{j=1}^{n} X_j \) and \( \lambda = \sum_{j=1}^{n} \lambda_j \).

(a) (10%) Show that \( T \) is distributed as \( P(\lambda) \).

(b) (10%) Find the conditional distribution of \( X_j \), given \( T = t \). [Note that \( P(\lambda) \) denotes the Poisson distribution with parameter \( \lambda \).]

5. Let X and Y be two independent random variables distributed as Beta(\( \alpha, \beta \)) and Beta(\( \alpha + \beta, \gamma \)), respectively. Set \( U = XY \) and \( V = X \).

(a) (10%) Find the joint p.d.f. of U and V.

(b) (10%) What is the marginal distribution of U?

Note that the p.d.f. of Beta(\( \alpha, \beta \)) is given by

\[ f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1, \quad \alpha, \beta > 0. \]

6. (15%) Let \( \{X_n\} \) be a sequence of random variables with \( P(X_n = \pm \frac{1}{n}) = \frac{1}{2} \).

Show that \( X_n \overset{a.s.}{\to} 0 \).