(1) (15%)  
(1a) Please define what is a compact set?  
(1b) Use (1a) to verify whether the following sets are compact or not?  
\[(0,1), \ (0,1), \ (0,1], \ [0,1].\]  
(1c) Please describe the open, connected and compact sets in \(\mathbb{R}\).  
(2) (15%) Prove or disprove there exists a continuous function  
\[f : \mathbb{R}^2 \to \mathbb{R}^2 \]  
such that  
\[f(A) = B\]  
where \(A\) and \(B\) are given by  
\[\begin{align*}  
\text{(2a)} & \quad y \xrightarrow{(x,y)} A \quad \text{in} \quad \mathbb{R}^2 \\
\text{(2b)} & \quad y \xrightarrow{(0,x)} A \quad \text{in} \quad \mathbb{R}^2 
\end{align*} \]  
(3) (10%) Given a set  
\[A = \left\{ x^{\alpha} \sin \frac{1}{x} \mid x \in (0,2\pi) \right\} \quad \alpha \geq 0 \]  
Is \(A\) a connected set? What is the closure of \(A\), i.e., \(\overline{A}\)?  
(4) (10%)  
(4a) Given a function  
\[f(x) = x^{\alpha} \quad x \in [0,1], \quad \alpha > 0.\]  
Prove or disprove it is uniform continuous?  
(4b) Given a function  
\[f(x) = x^{\alpha} \quad x \in [0,\infty], \quad \alpha > 0.\]  
Prove or disprove it is uniform continuous?  
(5) (15%) Given a sequence of trigonometric functions  
\[\left\{ \sin x \sin nx \mid x \in [0,2\pi] \right\} \quad n = 1,2,3\ldots\]  
Does it converge uniformly? What is the limit of the integral  
\[\int_0^{2\pi} \sin^2 nx f(x) dx \quad f \in C[0,2\pi]\]  
as \(n \to \infty\).  
(6) (20%) Given an improper integral  
\[\int_0^\infty \frac{\sin x}{x} dx\]  
Does it converge? Is it absolutely convergent? If converge please compute the integral.  
(7) (20%) Prove that the double improper integral  
\[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ax^2-2axy+y^2} \, dx \, dy \quad a > 0, \quad ac > b^2\]  
converges. Could You evaluate the integral? (You must explain every step rigorously.)