1. If \( x \sin x = \int_0^1 f(t) \, dt \) where \( f \) is a continuous function, find \( f(4) \). (10%)

2. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral. (12%)

3. A number \( x_0 \) is called a fixed point of a function \( f \) if \( f(x_0) = x_0 \).
   
   (a) Show that if \( f'(x) < 1 \) for all \( x \in \mathbb{R} \), then \( f \) has at most one fixed point. (10%)

   (b) Construct a function \( g \) such that \( g'(x) < 1 \) for all \( x \in \mathbb{R} \) and \( g \) has no fixed point. **Hint:** use \( \tan^{-1} x \). (10%)

4. If \( f(t) \) is continuous for \( t \geq 0 \), the Laplace transform of \( f \) is the function \( F \) defined by \( F(s) = \int_0^\infty f(t) e^{-st} \, dt \). Now suppose that \( 0 \leq f(t) \leq Me^{at} \) and \( 0 \leq f'(t) \leq Ke^{at} \) for \( t \geq 0 \), where \( f' \) is continuous. If the Laplace transform of \( f(t) \) is \( F(s) \), and the Laplace transform of \( f'(t) \) is \( G(s) \). Show that \( G(s) = sF(s) - f(0) \) for \( s > a \). (12%)

5. (a) Let \( a_1 = \sqrt{2} \) and \( a_{n+1} = \sqrt{2 + \sqrt{a_n}} \) for \( n = 1, 2, 3, \ldots \), show that \( \{a_n\} \) is convergent. (8%)

   (b) Prove that if \( a_n \geq 0 \) for all \( n \) and \( \sum_{n=1}^\infty a_n \) is convergent, then \( \sum_{n=1}^\infty \frac{\sqrt{a_n}}{n} \) is convergent. (8%)

6. Let \( T(x, y) = x^2 + xy + y^2 + x \) be the temperature function of the region \( \{(x, y) : x^2 + y^2 \leq 1, y \geq 0\} \). Find the maximal and the minimal temperature of this region. (15%)

7. Let \( R = \{(x, y) : x^2 + y^2 \leq 2, 0 \leq x \leq 1, y \geq 0\} \) and \( f(x, y) = \begin{cases} e^{x+y} & \text{if } x \leq y \\ 2e^{(1-y)} & \text{if } x > y. \end{cases} \)

   Evaluate the integral \( \int \int_R f(x, y) \, dA \). (15%)