1. Assume the temperature \( u(x, y) = \ln(x^2 + y^2) \) is a function of position \((x, y)\) for the region \( \{(x, y) | x^2 + y^2 \geq 1\} \). A particle moves in the region. The position of the particle at time \( t \) is \((x^*(t), y^*(t))\), where \( x^*(t) = t \cos(t) \) and \( y^*(t) = t \sin(t) \).

(a) Find the unit tangent vector \( T(t) \) and the unit normal vector \( N(t) \) for the path of the particle.

(b) Find the velocity and the acceleration of the particle.

(c) Find the temperature change rate at the particle, i.e. \( \frac{d}{dt} u(x^*, y^*) \).

(d) Find the gradient of the temperature at \((x, y)\), i.e. \( \nabla u(x, y) \).

(e) Show that the divergence of \( \nabla u \) is zero, i.e. \( \nabla \cdot \nabla u = 0 \).

(f) The heat flux is \( D \nabla u(x^*, y^*) \) where \( D \) is a positive constant. Find the total heat flow which diffuses from outside into the region.

2. Let \( \sum_{i=1}^{\infty} a_i \) be a real series.

(a) State the definition of "\( \sum_{i=1}^{\infty} a_i = L \)" (the series converges to \( L \)).

(b) Prove that "If \( \sum_{i=1}^{\infty} a_i \) converges then \( \lim_{i \to \infty} a_i = 0 \).

(c) Prove that "If \( \sum_{i=1}^{\infty} |a_i| \) converges then \( \sum_{i=1}^{\infty} a_i \) converges.

3. Let \( f(x) = \int_1^x \exp(t^3) \, dt \).

(a) Find \( f(1) \).

(b) Find \( f'(1) \).

(c) Find \( \int_0^1 xf(x) \, dx \).

4. Let \( f(x) \) be twice differentiable on an open interval containing \( a \) and \( b \), and \( \frac{d^2}{dx^2} f(x) < M \) for \( x \in [a, b] \). Let \( g(x) \) be the linear interpolation for \( f(x) \); i.e. \( g(x) = f(a) + \frac{f(b) - f(a)}{b-a} (x-a) \). Prove that \( |f(x) - g(x)| < \frac{|b-a|^3}{4} M \) for \( x \in [a, b] \).
5. Evaluate the following integrals

(a) \( \int_{0}^{\pi} \sin^2 x \, dx \)  
(b) \( \int_{-\infty}^{0} e^x \cos x \, dx \)  (improper integral)  
(c) \( \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \frac{y}{\sqrt{x^2 + y^2}} \, dy \, dx \)  (double integral)  

6. Evaluate the following limits

(a) \( \lim_{x \to \infty} x \sin \frac{1}{x} \)  
(b) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{(1 + \frac{i}{n})^n} \)  
(c) \( \lim_{x \to 0} \frac{\sin^{-1}(a + x) - \sin^{-1}(a - x)}{x} \)