Notations and Definitions:
- \( \mathbb{R}^n \): set of \( n \)-dimensional real vectors.
- \( \mathbb{R}^{n\times n} \): set of \( n \times n \) real matrices.
- \( \mathcal{P}_n(\mathbb{R}) \): set of real polynomials of degree \( \leq n \).
- \( A^T \): the transpose of the matrix \( A \).
- \( A \in \mathbb{R}^{n\times n} \) is positive definite if \( z^TAz > 0 \) for any nonzero \( z \in \mathbb{R}^n \).

Problems:
1. Let \( \mathcal{V} \) be an \( m \) (\( m \leq n \)) dimensional subspace of \( \mathbb{R}^n \), \( P \in \mathbb{R}^{n\times n} \) be a projection on \( \mathcal{V} \), that is, \( Px \in \mathcal{V} \) for any \( x \in \mathbb{R}^n \) and \( Pv = v \) for any \( v \in \mathcal{V} \).
   (i) Show that \( \text{det} \ P = 0 \). \( 10\% \)
   (ii) Let \( \{v_1, \ldots, v_m\} \) form an orthonormal basis of \( \mathcal{V} \). Find a project \( P \) on \( \mathcal{V} \) and represent \( P \) in a matrix form. \( 10\% \)
2. Let \( x_0 < x_1 < \cdots < x_n \) be \( n+1 \) distinct real numbers and \( y_k \in \mathbb{R} \), \( k = 0, 1, \ldots, n \). Show that there is a unique polynomial \( p(x) \in \mathcal{P}_n(\mathbb{R}) \) such that \( p(x_k) = y_k \), \( k = 0, 1, \ldots, n \). \( 10\% \)
3. Let \( A = A^T, B, D = D^T \in \mathbb{R}^{n\times n} \) and \( I \in \mathbb{R}^{n\times n} \) be the identity matrix.
   (i) Assume that \( A \) is positive definite. Show that if \( D - B^T A^{-1} B \) is positive definite, then \( M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \) is also positive definite. \( 10\% \)
   (ii) Verify if if \( \gamma > \|B\|_2^2 \), then \( M = \begin{bmatrix} I & B \\ B^T & \gamma I \end{bmatrix} \) is also positive definite. Here \( \|B\|_2^2 = \sup_{x \neq 0} \frac{x^T B^T B x}{x^T x} \). \( 10\% \)
4. Assume that \( A \in \mathbb{R}^{n\times n} \) is fixed. Let \( T \) be a linear operator on \( \mathbb{R}^{n\times n} \) defined by \( T(B) = AB \). Show that the minimal polynomial for \( T \) is the minimal polynomial for \( A \). \( 10\% \)
5. Let \( \mathcal{U} \) be an inner product space consisting of continuous complex-valued functions on the interval \( 0 \leq x \leq 1 \) with the inner product
   \[ (f|g) = \int_0^1 f(x) g(x) dx \] for any \( f, g \in \mathcal{U} \).
   (i) Show that \( h_k(x) = e^{2\pi ikx} \), \( k = \pm 1, \pm 2, \ldots \) are mutually orthogonal. Here \( i = \sqrt{-1} \). \( 5\% \)
   (ii) Verify the Bessel's inequality
   \[ \sum_{k=-n}^{n} \left| \int_0^1 f(t) e^{2\pi ikx} dt \right|^2 \leq \int_0^1 |f(t)|^2 dt \] for \( f \in \mathcal{U} \). \( 10\% \)

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6. Let 
\[ W = \{ f : [0, 1] \to \mathbb{R} \mid f \in C^2([0, 1]) \text{ and } f(0) = 0 = f(1) \} \]
be an inner product space with the inner product
\[ (f | g) = \int_0^1 f(x)g(x)dx \] 
for any \( f, g \in W \).

Here \( f \in C^2([0, 1]) \) means that \( f \) is defined on \([0, 1]\) and its second derivative is also defined and continuous on \([0, 1]\). Let \( D^2 \) be an operator on \( W \) defined by
\[ D^2(f) = \frac{d^2f}{dx^2} \] 
for \( f \in W \).

(i) Show that \( D^2 \) is self-adjoint. 10% (Hint: Use integration by parts!)

(ii) Show that \( D^2 \) is positive definite, i.e., \( (D^2f|f) > 0 \) for any nonzero function \( f \in W \). 10% 

7. Let \( T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R}) \) be defined by \( T(f) = f(0) + f(1)(x + x^2) \). Show that \( T \) is diagonalizable. 10%