1. Let $X$ and $Y$ be random variables and let $A$ be an event. Prove that the function
\[
Z(w) = \begin{cases} 
X(w), & \text{if } w \in A \\
Y(w), & \text{if } w \in A^c
\end{cases}
\]
is a random variable. (7%)

2. Let $X$ and $Y$ be i.i.d. with continuous distribution function $F$. Find the probabilities $P(X = Y)$ and $P(X < Y)$. (8%)

3. Let $X_1, X_2$ be independently distributed as $N(\mu, \sigma^2)$, $\sigma > 0$, $i = 1, 2$, and let
\[
\begin{cases}
Z_1 = X_1 \cos \theta + X_2 \sin \theta \\
Z_2 = X_2 \cos \theta - X_1 \sin \theta.
\end{cases}
\]
Find the correlation coefficient between $Z_1$ and $Z_2$, and show that
\[
0 \leq \rho^2 \leq \left(\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^2,
\]
where $\rho$ denotes the correlation coefficient of $Z_1$ and $Z_2$. (10%)

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of events such that $\lim_{n \to +\infty} A_n = A$ ($\in \mathcal{F}$).
   (a) Show that $\lim_{n \to +\infty} P(A_n) = P(A)$. (15%)
   (b) Prove that if $\sum_{n=1}^{+\infty} P(A_n) < +\infty$, then $P\left(\bigcap_{k=1}^{+\infty} \bigcup_{n=k}^{+\infty} A_n\right) = 0$. (10%)

5. Let $X_1, \ldots, X_n$ be independently distributed as $N(\mu, \sigma^2)$, $\sigma > 0$.
   (a) Prove that $\bar{X}$ and $Y = (X_1 - \bar{X}, \ldots, X_n - \bar{X})'$ are independent. (10%)
   (b) Prove that $\frac{nS^2}{\sigma^2}$ is distributed as $\chi^2_{n-1}$, where $S^2 = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X})^2$. (10%)

6. (a) Let $X, Y$ be random variables on the probability space $(\Omega, \mathcal{F}, P)$. Assume that $p, q > 1 \implies \frac{1}{p} + \frac{1}{q} = 1$ and $E|X|^p < +\infty$, $E|Y|^q < +\infty$. Prove that $E|XY| \leq (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}$. (15%)
   (b) Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables and let $X$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Prove that if $X_n \xrightarrow{q.m.} X$, then
   (i) $EX_n \xrightarrow{n \to +\infty} EX$,
   (ii) $EX_n^2 \xrightarrow{n \to +\infty} EX^2$, and hence $\text{Var}(X_n) \xrightarrow{n \to +\infty} \text{Var}(X)$.
   (Note. $X_n \xrightarrow{q.m.} X$ means that $\{X_n\}_{n \in \mathbb{N}}$ converges to $X$ in quadratic mean as $n \to +\infty$.)