1. (15 pts.) Suppose \( \{a_n\}_{n \in \mathbb{N}} \) is a sequence of positive numbers. Show that

\[
\lim_{n \to +\infty} \sqrt[n]{a_n} \leq \lim_{n \to +\infty} \frac{a_{n+1}}{a_n}.
\]

2. Suppose \( f : [a, b] \to \mathbb{R} \) is a \( C^1 \) injection.
   (a). (7 pts.) Show that \( \int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a) \).
   (b). (7 pts.) If \( f(x) \geq 0, \forall x \in [a, b] \), give a geometric interpretation for the formula in (a).
   (c). (6 pts.) Evaluate \( \int_0^1 \left( (x - 1)\frac{1}{2} + 1 \right)^{\frac{1}{2}} dx \).

3. (15 pts.) Suppose \( E \) is a nonempty compact subset of \( \mathbb{R}^n \) and \( f, g : \mathbb{R}^n \to \mathbb{R} \) are \( C^1 \) such that \( f = g \) on the boundary of \( E \). Show that there is a point \( x_0 \in E \) such that \( \nabla f(x_0) = \nabla g(x_0) \).

4. (15 pts.) If \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) uniformly on every closed subinterval of \( (0, 1) \), does it follow that \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) uniformly on \( (0, 1) \)? Support your statement with either a proof or a counterexample.

5. (a). (8 pts.) State the Implicit Function Theorem.
   (b). (7 pts.) Decide whether it is possible to solve the pair of equations

\begin{align*}
xy^2 + xzu + yv^2 - 3 &= 0 \\
u^2yz + 2xu - u^2v^2 - 2 &= 0
\end{align*}

for \( u \) and \( v \) as \( C^1 \) functions of \( (x, y, z) \) in a neighborhood of the points \( (u, v) = (1, 1) \) and \( (x, y, z) = (1, 1, 1) \).

6. For any \( n \in \mathbb{N} \), let \( a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \).
   (a). (10 pts.) Show that \( \{a_n\}_{n \in \mathbb{N}} \) is convergent to \( \gamma \) for some \( \gamma \in \mathbb{R} \).
   (b). (10 pts.) Express \( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \) as \( 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \gamma + \ln n + \varepsilon_n \) to evaluate

\[
\sum_{k=1}^{+\infty} \frac{1}{k(2k-1)}.
\]