1. Let $T$ be a linear transformation form the vector space $V$ into the vector space $W$ and $v_1, v_2, \ldots, v_n$ be a basis for $V$. Prove that
   (i) $T$ is onto if and only if $T(v_1), T(v_2), \ldots, T(v_n)$ span $W$. (6%)
   (ii) $T$ is one-to-one if and only if $T(v_1), T(v_2), \ldots, T(v_n)$ is linearly independent. (8%)

2. Let $A, B \in M_{n \times n}(\mathbb{R})$, show that
   (i) if $A$ and $B$ are upper triangular, than $AB$ is alos upper triangular; (5%)
   (ii) $\text{rank } AB = \text{rank } BA$ is not always true; (5%)
   (iii) $\text{rank } A \leq 1$ if and only if there exist $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ such that $A = x^t y$. (10%)

3. Suppose $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$, $T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a + d & a + b + c + d \\ 0 & -a - d \end{bmatrix}$. (8%)
   (i) Find $\ker (T)$, $\text{Im } (T)$, nullity $(T)$ and rank $(T)$. (10%)
   (ii) Find the characteristic polynomial and the minimal polynomial of $T$. (4%)

4. Let $A = \begin{bmatrix} 6 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 5 & 1 \\ \sqrt{2} & 1 & 5 \end{bmatrix}$. (10%)
   (i) Find an orthogonal matrix $Q$ such that $Q^{-1}AQ$ is diagonal.
   (ii) Find a positive-definite matrix $B$ such that $B^2 = A$. (5%)
   (iii) Evaluate $\lim_{n \to \infty} \left( \frac{1}{3} A \right)^n$. (5%)

5. Let $V$ be a finite-dimensional inner product space and $T : V \rightarrow V$ be linear.
   Prove the following statements:
   (i) If the minimal polynomial of $T$ is $m(t) = p(t)q(t)$ where $p(t)$ and $q(t)$ are relative prime polynomials, then $V = \ker (p(T)) \oplus \ker (q(T))$. (10%)
   (ii) If $T$ is idempotent (i.e. $T^2 = T$), then $V = \ker (T) \oplus \text{Im } (T)$. (4%)
   (iii) If $T$ is idempotent, then $T$ is self-adjoint if and only if $\ker (T) \perp \text{Im } (T)$. (10%)