Ordinary Differential Equation

1. Solve the following differential equations: (20%)
   (a) \( \frac{dy}{dx} = \frac{2x+y}{y^2+1} \)
   (b) \( \frac{dy}{dx} = \frac{y+2x+2}{z+y} \)
   (c) \( x^2y'' = y'(3x-2y') \)
   (d) \( y'' - y' - 2y = 4x^2 \)

2. Study the asymptotic behavior of the differential equation (10%)
   \[ y'(x) = (y-1)(y-2)(y-3) \quad y(0) = c \in \mathbb{R} \]
   as \( x \to \infty \) without solving the equation. You need to separate the range of \( c \). Can you sketch the integral curve?

3. Use the method of variation of parameters to show that (20%)
   \[ y(x) = c_1 \cos x + c_2 \sin x + \int_0^x f(\xi) \sin(x-\xi)d\xi \]
   is a general solution to the 2nd order differential equation
   \[ y'' + y = f(x) \]
   where \( f(x) \) is a continuous function on \((-\infty, \infty)\).

4. Find the solution to the initial value problem (20%)
   \[ x'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} 2e^{5t} \\ 2e^{2t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

5. Apply the Laplace transform to solve the initial value problem (15%)
   \[ y'' + 2ty' - 4y = 1, \quad y(0) = y'(0) = 0 \]

6. The Bessel function is defined by (15%)
   \[ J_\nu(x) \equiv \frac{1}{\pi} \int_0^\pi \cos(\nu t - x \sin t)dt \quad -\infty < x < \infty, \quad \nu \in \mathbb{R} \]
   where \( f \) is a continuous and differentiable function. Show that when \( \nu \) is an integer then it satisfies the Bessel equation
   \[ x^2J''_\nu(x) + xJ'_\nu(x) + (x^2 - \nu^2)J_\nu(x) = 0 \]

\( \overline{F 801 3-2} \)