1. The sales of a convenience store on a randomly selected day are \( X \) thousand dollars, where \( X \) is a random variable with a distribution function of the following form:

\[
P(z) = \begin{cases} 
0, & z < 0 \\
\frac{1}{2} z^2, & 0 \leq z < 1 \\
(4z - z^2), & 1 \leq z < 2 \\
1, & z \geq 2.
\end{cases}
\]

Suppose that this convenience store's total sales on any given day are less than $2000.

(a) Find the value of \( k \). (5%)

(b) Let \( A \) and \( B \) be the events that tomorrow the store's total sales are between 500 and 1500 dollars, and over 1000 dollars, respectively. Find \( P(A) \) and \( P(B) \). (10%)

(c) Are \( A \) and \( B \) independent events? (5%)

2. Let \((X, Y)\) be a continuous random vector with the probability density function

\[
f(x, y) = \begin{cases} 
4x(1-y), & 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

(a) Find \( E(X^2Y^4) \), \( j, k \in \mathbb{Z}_+ \cap \{0\} \). (10%)

(b) Find \( \text{Var}(X - Y) \) and \( \rho(X, Y) \) (the correlation coefficient of \( X \) and \( Y \)). (10%)

3. Suppose that \( X, Y \in L^2 \).

(a) Show that

\[ \text{Var}(X) = \text{Var}[E(X|Y)] + E[\text{Var}(X|Y)] \]

where \( \text{Var}(X|y) = E[(X - E(X|y))^2|y] \). (10%)

(b) For each \( \theta \in [0, 2\pi] \), define

\[ X_\theta = X \cos \theta - Y \sin \theta \]
\[ Y_\theta = X \sin \theta + Y \cos \theta. \]

Show that there is at least one value of \( \theta \) for which \( X_\theta \) and \( Y_\theta \) are uncorrelated. (10%)

4. Let \( f(x, y) \) be the joint probability density function of continuous random variables \( X \) and \( Y \);

\[ f(x, y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1 - \rho^2}} \exp \left( -\frac{1}{2(1 - \rho^2)} q(x, y) \right), \quad (x, y) \in \mathbb{R}^2, \]

where \( \rho \) is the correlation coefficient of \( X \) and \( Y \) and

\[ q(x, y) = \left( \frac{z - \mu_X}{\sigma_X} \right)^2 - 2 \rho \left( \frac{z - \mu_X}{\sigma_X} \right) \left( \frac{y - \mu_Y}{\sigma_Y} \right) + \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 \]

\( (\mu_X, \mu_Y \in \mathbb{R}, \sigma_X, \sigma_Y > 0, -1 < \rho < 1). \)

(a) Find the conditional distribution of \( Y \), given \( X = z \) (\( z \in \mathbb{R} \)). (10%)

(b) For what values of \( \alpha \) is the variance of \( \alpha X + Y \) minimum? (5%)

(c) Show that if \( \sigma_X = \sigma_Y \), then \( X + Y \) and \( X - Y \) are independent random variables. (5%)

5. Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of i.i.d. r.v.'s with common probability density function

\[ f(x) = \begin{cases} 
e^{-\frac{(x-\theta)}{\theta}}, & x \geq \theta \\
0, & \text{otherwise}
\end{cases} \]

Write \( \overline{X}_n = \sum_{i=1}^{n} X_i / n \), \( X_{(1)} = \min \{X_1, \ldots, X_n\} \).

(a) Show that \( \overline{X}_n \xrightarrow{p} 1 + \theta \). (10%)

(b) Show that \( X_{(1)} \xrightarrow{p} \theta \). (10%)