1. Let \( M_2(\mathbb{R}) \) be the set of all \( 2 \times 2 \) real matrices and let
\[
L = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid a + d = 0 \right\}.
\]
(a) (5%) Show that \( L \) is a subspace of \( M_2(\mathbb{R}) \).
(b) (5%) Find the dimension of \( L \).

2. Let \( S : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear transformation such that the matrix of \( S \) with respect to the standard basis is given by
\[
\begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
(a) (5%) Show that \( S \) is an isomorphism.
(b) (5%) Find the matrix of \( S \) with respect to the basis \( \{(1,1,0), (0,1,-1), (1,1,1)\} \).

3. (10%) Let \( V \) be a vector space over \( \mathbb{R} \). Let \( \alpha : V \to V \) be a linear transformation such that \( \alpha^3 = \alpha \). Show that \( V = W_0 \oplus W_1 \oplus W_{-1} \), where \( W_0 = \ker \alpha \), \( W_1 = \{ v \in V \mid \alpha(v) = v \} \) and \( W_{-1} = \{ v \in V \mid \alpha(v) = -v \} \).

4. Let \( \{v_1, v_2, v_3, v_4\} \) be a basis of a vector space \( V \) over a field \( F \). Determine if the following set is a basis of \( V \). Justify your answer.
(a) (5%) \( \{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\} \).
(b) (5%) \( \{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1\} \).

5. (10%) Let \( A \) be an \( n \times n \) matrix over \( \mathbb{R} \). Suppose that \( \text{tr} A^k = 0 \) for all positive integers \( k \). Is \( A = 0 \)? Justify your answer.

6. Let \( X_1 \) and \( X_2 \) be subspaces of a finite dimensional vector space \( V \) of dimension \( n \).
(a) (7%) Suppose that both \( X_1 \) and \( X_2 \) are both of dimension \( n - 1 \) and \( X_1 \neq X_2 \). What is the dimension of \( X_1 \cap X_2 \)? Justify your answer.
(b) (3%) Let \( X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0 \} \) and \( Y = \{(x, y, x, y) \in \mathbb{R}^4 \mid x, y \in \mathbb{R} \} \) be subspaces of \( \mathbb{R}^4 \). What is the dimension of \( X \cap Y \)?
7. (10 %) Let $V^* = \{ f : V \rightarrow F | f \text{ is linear} \}$ be the dual space of $V$. For any linear transformation $T : V \rightarrow V$, we define a linear transformation $T^* : V^* \rightarrow V^*$ by

$$T^*(f) = f \circ T \quad \text{for any } f \in V^*.$$ 

Suppose that the matrix of $T$ with respect to a basis $B = \{ x_1, x_2, \ldots, x_n \}$ is given by $A = (a_{ij})_{1 \leq i, j \leq n}$. Show that the matrix of $T^*$ with respect to the dual basis $B^* = \{ x_1^*, x_2^*, \ldots, x_n^* \}$ is the transpose of $A$.

8. (15 %) Let $V$ be an $n$-dimensional vector space over a field $F$ and let $f : V \rightarrow V$ be a linear transformation. Suppose that the minimal polynomial of $f$ is given by $(x - \lambda)^n$ for some $\lambda \in F$. Show that there is a basis $\{v_1, \ldots, v_n\}$ of $V$ such that

$$f(v_1) = \lambda v_1 \quad \text{and} \quad f(v_i) \in \text{span}\{v_{i-1}, v_1\}, \quad i = 2, \ldots, n.$$ 

9. (15 %) Show that for any real numbers $a, b, c, d$,

$$\begin{vmatrix} a^2 & (a + 1)^2 & (a + 2)^2 & (a + 3)^2 \\ b^2 & (b + 1)^2 & (b + 2)^2 & (b + 3)^2 \\ c^2 & (c + 1)^2 & (c + 2)^2 & (c + 3)^2 \\ d^2 & (d + 1)^2 & (d + 2)^2 & (d + 3)^2 \end{vmatrix} = 0.$$ 

The End