Simultaneously Diagonalization

Definition 0.1. Two linear operators $T$ and $U$ on a finite-dimensional vector space $V$ are called **simultaneously diagonalizable** if there exists an ordered basis $\beta$ such that both $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Similarly, $A, B \in M_{n \times n}(F)$ are called **simultaneously diagonalizable** if there exists an invertible matrix $Q \in M_{n \times n}(F)$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

1. Let $A, B$ are two diagonalizable $n \times n$ matrices. Prove that $A, B$ is simultaneously diagonalizable iff $A, B$ commute (i.e. $AB = BA$).

2. Let $S = \{T_1, \cdots, T_k\}$ be a set of diagonalizable linear operator on a vector space $V$. Assume that $T_iT_j = T_jT_i$, for all $1 \leq i, j \leq k$. Prove that there exists a basis $\mathcal{B}$ such that $[T_i]_\mathcal{B}$ are diagonal for all $1 \leq i \leq k$. 


3. Let \( A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix} \). Find a basis which simultaneously diagonalizes \( A \) and \( B \). [Answer: \{(1,1),(1,2)\}]

4. Prove that if \( T \) is a diagonalizable linear operator on a finite-dimensional vector space \( V \), and let \( m \) be any positive integer. Prove that \( T \) and \( T^m \) are simultaneously diagonalizable.