Linear Algebra I, Final, Yung-fu Fang, 2010/06/21

1. Let $A$ be an $n \times n$ matrix. Show that $\text{rank}(A') = \text{rank}(A)$. [10%]

2. Determine whether the matrix $A = \begin{bmatrix} 0 & 4 & 8 \\ 4 & 8 & 4 \\ 6 & 6 & 2 \end{bmatrix}$ is invertible, and if it is, compute its inverse. Use the method given in the textbook. [10%]

3. Solve the system of linear equations:
   
   $\begin{align*}
   2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\
   x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\
   x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\
   2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14,
   \end{align*}$

   following procedure. First write down the corresponding matrix equation, $Ax = b$, and the augmented matrix $(A|b)$. Then use the Gaussian elimination to transform the matrix $A$ into its reduced row echelon form. Finally solve the system of linear equations corresponding to the last matrix. [10%]

4. Let $V = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 : x_1 + 7x_2 + 5x_3 - 4x_4 + 2x_5 = 0\}$. Show that $V$ is a subspace of $R^5$. Show that $S = \{(-2, 0, 0, -1, -1), (1, 1, -2, -1, -1), (-5, 1, 0, 1, 1)\}$ is a linearly independent subset of $V$. Find a basis $\beta$ for $V$. Extend $S$ to a basis for $V$. [20%]

5. Prove that the determinant of an upper triangular $n \times n$ matrix is the product of its diagonal entries. [10%]

6. Prove that if $E$ is an elementary matrix, then $\det(E') = \det(E)$. [15%]

7. Let $M = \begin{bmatrix} 1 & c_0 & c_0^2 & c_0^3 & c_0^4 \\ 1 & c_1 & c_1^2 & c_1^3 & c_1^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_4 & c_4^2 & c_4^3 & c_4^4 \end{bmatrix}$. Prove that $\det(M) = \prod_{0 \leq i < j \leq 4} (c_j - c_i)$, the product of all terms of the form $c_j - c_i$ for $0 \leq i < j \leq 4$. Hint: Use elementary row operations. [15%]

8. Let $\delta : M_{2\times2}(F) \to F$ be a function with the following 3 properties. (i) $\delta$ is a linear function of each row of the matrix when the other row is held fixed. (ii) If the 2 rows of $A \in M_{2\times2}(F)$ are identical, then $\delta(A) = 0$. (iii) If $I$ is the $2 \times 2$ identity matrix, then $\delta(I) = 1$. Prove that $\delta(A) = \det(A)$ for all $A \in M_{2\times2}(F)$. [10%]

9. Let $V = P_1(R)$ and $W = R^2$ with respective standard ordered bases $\beta$ and $\gamma$. Let $W^* \equiv L(W, R)$ and $V^* \equiv L(V, R)$ Define $T : V \to W$ by $T(p(x)) = (p(0) - 2p(1), p(0) + p'(0))$, where $p'(x)$ is the derivative of $p(x)$. Define $T^i : W^* \to V^*$ by $T^i(g) = gT$ for all $g \in W^*$. Let $\beta^*$ and $\gamma^*$ be the dual bases of $\beta$ and $\gamma$ respectively.
   
   (a). For $f \in W^*$ defined by $f(a, b) = a - 2b$, compute $T^i(f)$.
   (b). Compute $[T^i]^{\beta^*}_{\gamma^*}$ without appealing to Thm 2.25.
   (c). Compute $[T]^{\beta}_{\gamma}$ and its transpose. [15%]

Have A Nice Summer!