1. Show that the set of $2 \times 3$ matrices is a vector space. \hfill [10%]

2. Show that there is a unique function $g(x) = Ax^2 + Bx + C$ that passes through the 3 points, $(c_0, b_0)$, $(c_1, b_1)$, and $(c_2, b_2)$, of which $c_0$, $c_1$, and $c_2$ are different, by the following approach: \hfill [20%]
   (a) Solve the system of the equations for the coefficients $A$, $B$, and $C$ to get the existence of such function $g$.
   (b) Rewrite $g(x) = Ax^2 + Bx + C$ into the form of $g(x) = b_0 f_0(x) + b_1 f_1(x) + b_2 f_2(x)$, where $f_0$, $f_1$, and $f_2$ are the Lagrange polynomials.
   (c) Show that $P_2(R)$ is a 3-dimensional vector space and $\{f_0, f_1, f_2\}$ is a basis for $P_2(R)$.
   (d) Show that the expression $g(x) = Ax^2 + Bx + C$, whose graph (parabola or line) contains the given three points.

3. Let $V$ be a vector space. Determine all linear transformations $T : V \to V$ such that $T = T^2$. \hfill [10%]

4. Suppose that $W$ is $T$-invariant. Prove that $N(T_W) = N(T) \cap W$ and $R(T_W) = T(W)$. \hfill [10%]

5. Let $T : P_3(R) \to P_2(R)$ defined by $T(f(x)) = f'(x)$ and $U : P_2(R) \to P_3(R)$ defined by $U(f(x)) = \int_0^x f(t)dt$. Let $\beta$ and $\gamma$ be the standard ordered bases for $P_3(R)$ and $P_2(R)$ respectively. Find the matrices $[T]_{\beta}^{\gamma}$, $[U]_{\gamma}^{\beta}$, $[TU]_{\gamma}^{\gamma}$, and $[UT]_{\gamma}^{\gamma}$. Also Compute $[U]_{\beta}^{\beta}$, $[T]_{\beta}^{\beta}$, and $[TU]_{\beta}^{\beta}$. \hfill [15%]

6. Let $T : R^2 \to R^2$ be the rotation around the origin by the angle $\theta$ in the counterclockwise direction. Let $\beta = \{e_1, e_2\}$ be the standard basis for $R^2$. Show that $T$ is a linear transformation. Find $[T]_{\beta}$. \hfill [10%]

7. State and prove the Dimension theorem. \hfill [15%]

8. Let $A$ be a $2 \times 2$ matrix. Let $\lambda \in R$ and $\xi \in R^2$ such that $(A - \lambda I)\xi = 0$. Suppose that the null space of $A - \lambda I$ is 1-dimensional and $(A - \lambda I)^2$ is a zero matrix. Show there is a vector $\eta \in R^2$ such that $(A - \lambda I)\eta = \xi$. Show that $\beta = \{\xi, \eta\}$ is an ordered basis for $R^2$. Find $[L_A]_{\beta}$. \hfill [10%]

9. Let $V$ be a finite dimensional vector space. State the definitions of $V^*$ and $V^{**}$. Show that $\dim(V) = \dim(V^*) = \dim(V^{**})$. \hfill [10%]