Show All Work

1. (a) State the test for diagonalization. [5%]
    (b) State the Cayley-Hamilton Theorem. [5%]
    (c) State the Gram-Schmidt Process. [5%]
    (d) State the Schur Theorem. [5%]

2. Let $T : P_2(R) \to P_2(R)$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$. Show that $T$ is diagonalizable and find the matrices $Q$ and $D$ such that $Q^{-1}AQ = D$. [10%]

3. Let $T : R^2 \to R^2$ be the rotation by $\theta$. Prove that $T$ is a linear operator. Is $T$ diagonalizable? Explain! [10%]

4. Let $B_1 \in M_{k\times k}(F)$, $B_2 \in M_{k\times(n-k)}(F)$, and $B_3 \in M_{(n-k)\times(n-k)}(F)$. Show that
   \[
   \det \begin{pmatrix} B_1 - tI_k & B_2 \\ 0 & B_3 - tI_{n-k} \end{pmatrix} = \det (B_1 - tI_k) \det (B_3 - tI_{n-k}).
   \] [10%]

5. Let $T$ be a linear operator on a finite-dimensional vector space $V$, and let $W$ be a $T$-invariant subspace of $V$. Define $T : V/W \to V/W$ by $T(v + W) = T(v) + W$ for any $v + W \in V/W$. Show that if both $T_W$ and $T$ are diagonalizable and have no common eigenvalues, then $T$ is diagonalizable. [10%]

6. Let $V$ be a finite-dimensional inner product space with an orthonormal ordered basis $\beta = \{v_1, \cdots, v_n\}$, $T$ a linear operator on $V$, and the matrix $A = [T]_\beta$. Prove that, for all $i$ and $j$, $A_{ij} = \langle T(v_j), v_i \rangle$. Give a direct proof. [10%]

7. Let $\| \cdot \|$ be a norm on a real vector space $V$ satisfying the parallelogram law,
   \[
   \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.
   \]
   Define
   \[
   \langle x, y \rangle = \frac{1}{4} [\|x + y\|^2 - \|x - y\|^2].
   \]
   Show that
   (a) $\langle x, 2y \rangle = 2 \langle x, y \rangle$, for all $x, y \in V$. [5%]
   (b) $\langle x + u, y \rangle = \langle x, y \rangle + \langle u, y \rangle$, for all $x, u, y \in V$. [5%]

8. Let $A \in M_{m\times n}(F)$ and $b \in F^m$. Suppose that the system of equations $Ax = b$ is consistent.
   (a) Prove that $R(L_A^*) = N(L_A)$. [5%]
   (b) Prove that the minimal solution $s$ to $Ax = b$ is in $R(L_A^*)$. [5%]
   (c) Find the minimal solution to
   \[
   \begin{align*}
   x + 2y - z &= 1, \\
   2x + 3y + z &= 2, \\
   4x + 7y - z &= 4.
   \end{align*}
   \] [5%]

9. Let $T$ be a normal operator on a finite-dimensional real inner product space $V$ whose characteristic polynomial splits. Show that $V$ has an orthonormal basis of eigenvectors of $T$. Hence that $T$ is self-adjoint. [10%]