Theorem 4. (Arzela-Ascoli Theorem). Let $K$ be a compact subset of $\mathbb{R}^p$ and let $\mathcal{F}$ be a subset of the set of continuous functions on $K$, i.e. $\mathcal{F} \subset \{f : K \to \mathbb{R}^q \mid f$ is continuous on $K\}$. Then the following properties are equivalent:

1. The family $\mathcal{F}$ is bounded and uniformly equicontinuous on $K$.
2. Every sequence from $\mathcal{F}$ has a subsequence which is uniformly convergent on $K$ to a uniformly continuous function $f$ (which may not belong to $\mathcal{F}$).

Proof of (2) $\Rightarrow$ $\mathcal{F}$ is bounded: Suppose that $\mathcal{F}$ is not bounded, then there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ such that $\|f_n\|_K \geq n$, i.e. $\exists x_n \in K$ such that $\|f_n(x_n)\| \geq n$. This implies that $f_n$ cannot have any subsequence that converges uniformly on $K$, since if $f_n$ had a subsequence, still denoted by $f_n$, that converges uniformly on $K$ to a uniformly continuous function $f$ (which may not belong to $\mathcal{F}$), then we have $\lim_{n \to \infty} \|f_n\|_K \leq \lim_{n \to \infty} \|f_n - f\|_K + \lim_{n \to \infty} \|f\|_K < \infty$ which is a contradiction.

Proof of (2) $\Rightarrow$ $\mathcal{F}$ is uniform equicontinuous: Suppose that $\mathcal{F}$ is not uniformly equicontinuous on $K$, then there exists an $\epsilon_0 > 0$, such that for each $n \in \mathbb{N}$, there exist $f_n \in \mathcal{F}$, and $x_n, y_n \in K$ satisfying that $\|x_n - y_n\| < \frac{1}{n}$ and $\|f_n(x_n) - f_n(y_n)\| \geq \epsilon_0$. This implies that $f_n$ cannot have any subsequence that converges uniformly on $K$, since if $f_n$ had a subsequence, still denoted by $f_n$, that converges uniformly on $K$ to a uniformly continuous function $f$ (which may not belong to $\mathcal{F}$), thus

$c < \epsilon_0 \leq \lim_{n \to \infty} \|f_n(x_n) - f_n(y_n)\| = \lim_{n \to \infty} \|f_n(x_n) - f(x_n) + f(x_n) - f(y_n) + f(y_n) - f_n(y_n)\|
\leq \lim_{n \to \infty} \|f_n(x_n) - f(x_n)\| + \lim_{n \to \infty} \|f(x_n) - f(y_n)\| + \lim_{n \to \infty} \|f_n(y_n) - f(y_n)\|
\leq \lim_{n \to \infty} \|f_n - f\|_K + \lim_{n \to \infty} \|f(x_n) - f(y_n)\| + \lim_{n \to \infty} \|f_n - f\|_K = 0,
which is a contradiction. Note that, in the last equality,

$\lim_{n \to \infty} \|f_n - f\|_K = 0$ since $f_n$ converges to $f$ uniformly on $K$, and

$\lim_{n \to \infty} \|f(x_n) - f(y_n)\| = 0$ since $f$ is uniformly continuous on $K$, and $\|x_n - y_n\| < \frac{1}{n}$.

Proof of (1) $\Rightarrow$ (2) Step (1): Using diagonal process to extract a subsequence from a given sequence $\{f_n\} \subset \mathcal{F}$ (a bounded family): Let $S = K \cap \mathbb{Q}^p = \{x_i\}_{i \in \mathbb{N}}$. Note that $S$, called a dense subset of $K$, is a countable set, and $\mathcal{F} = K$, i.e. for each $x \in K$, and for each $\delta > 0$, there exists $x_i \in S$ such that $x_i \in B(\delta, x) = \{z \in \mathbb{R}^p \mid \|z - x\| < \delta\}$.

Suppose that $\mathcal{F}$ is bounded and $\{f_n\}$ is any sequence in $\mathcal{F}$, then, since

(*) $\{f_n(x_i)\}$ is bounded in $\mathbb{R}^q \Rightarrow \{f_n\}$ has a subsequence, denoted $\{f_1^n\}$, converges at $x_1$.

Next, since

(*) $\{f_2^n(x_2)\}$ is bounded in $\mathbb{R}^q \Rightarrow \{f_1^n\}$ has a subsequence, denoted $\{f_2^n\}$, converges at $x_2$, and $x_1$ (since $\{f_2^n\}$ is a subsequence of $\{f_1^n\}$ that converges at $x_1$).

Inductively, for each $k \geq 2$, since

(*) $\{f_k^n(x_{k+1})\}$ is bounded in $\mathbb{R}^q \Rightarrow \{f_k^n\}$ has a subsequence, denoted $\{f_k^{k+1}\}$, converges at $x_{k+1}$, and $x_j$ (since $\{f_k^{k+1}\}$ is a subsequence of $\{f_k^n\}$ that converges at $x_j$ for each $j = 1, \ldots, k.$)
Proof of (1) ⇒ (2) Step (2): Using the uniform equicontinuity of $\mathscr{F}$ to show that the subsequence $g_n$ (of $f_n$) converges uniformly to $f$ on $K$: In particular, we shall show that the sequence $g_n$ (which is a subsequence of $\{f_n\} \subset \mathscr{F}$) satisfies the Cauchy criterion for uniform convergence on $K$ [to a uniformly continuous function $f$] (which may not belong to $\mathscr{F}$), i.e. for each $\epsilon > 0$ and for each $x \in K$, there exists $L \in \mathbb{N}$ such that if $m, n \geq L$, then $\|g_n(x) - g_m(x)\| < \epsilon$.

For each $\epsilon > 0$, since $\mathscr{F}$ is uniformly equicontinuous on $K$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in K$ satisfying that $\|x - y\| < \delta$, then $\|h(x) - h(y)\| < \epsilon$ for all $h \in \mathscr{F}$. This implies that for each each $x \in K$, since $\bar{S} = K \cup \bigcup_{i=1}^{\infty} B(\delta, x_i)$, and each $g_n \in \mathscr{F}$, there exists an $x_i \in S$ such that

\[ (*) \|x - x_i\| < \delta \implies \|g_n(x) - g_n(x_i)\| < \frac{\epsilon}{3} \text{ for all } n \in \mathbb{N}. \]

Also, for the same $\epsilon > 0$, since $\lim_{n \to \infty} g_n(x_i) = f(x_i)$, there exists $L \in \mathbb{N}$ such that

(†) for any $m, n \geq L$, we have $\|g_m(x_i) - g_n(x_i)\| < \frac{\epsilon}{3}$.

Therefore, combining (⋆) and (†), we have for each $x \in K$ and for any $m, n \geq L$, we have

$\|g_n(x) - g_m(x)\| \leq \|g_n(x) - g_n(x_i)\| + \|g_n(x_i) - g_m(x_i)\| + \|g_m(x_i) - g_m(x)\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

This implies that for any $m, n \geq L$, $\|g_n - g_m\|_K < \epsilon$, and, $g_n$ converges uniformly on $K$ [to a uniformly continuous function $f$] (which may not belong to $\mathscr{F}$).
Study guide of Chapter 3:
Outlines of (3.1): Apply the Implicit Function Theorem to study the following question:

Q: Let \( W \) be an open subset of \( \mathbb{R}^m \times \mathbb{R}^n, F: W \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \) be a (vector-valued) function of class \( C^1 \), and \((a, b)\) be a point in \( W \subset \mathbb{R}^m \times \mathbb{R}^n \) satisfying \( F(a, b) = 0 \). When is there
(1) a function \( f(x) \), defined in some open set in \( \mathbb{R}^n \) containing \( a \), and
(2) an open set \( U \subset W \subset \mathbb{R}^{m+n} \) containing \((a, b)\),
such that for \((x, y) \in U,\)
\[
F(x, y) = 0 \iff y = f(x)\?
\]

Read e.g. Theorem 3.1, Theorem 3.9 in the book, and Exercises 1 – 3, 5 – 7, 9 in (3.1)

Outlines of (3.2), (3.3), (3.5): There are three common ways of locally representing smooth \( k \)-dimensional manifolds (e.g. curves, or surfaces if \( k = 1 \) or 2.) in \( \mathbb{R}^n \):

Type (1): as the graph of a function, \( y = f(x) \), where \( f \) is of class \( C^1 \), \( x \in B \), and \( B \) is a connected open set in \( \mathbb{R}^k \);
Type (2): as the locus of an equation \( F(x, y) = 0 \), where \( F \) is of class \( C^1 \);
Type (3): parametrically, as the range of a \( C^1 \) function \( g : U \subset \mathbb{R}^m \to \mathbb{R}^n \) whose differential \( Dg \) has rank \( k \) everywhere in a connected open set \( U \).

Note that if a manifold can be represented locally in Type (1), then it can be represented in both Types (2) and (3) by setting \( F(x, y) = y - f(x) \), and \( g(x) = (x, f(x)) \) for \( x \in B \subset \mathbb{R}^k \), respectively. It is natural to ask the following converse, i.e.

Q: When can a manifold given locally in Types (2) or (3) be represented in Type (1)?

Read e.g. Theorem 3.11, Theorem 3.15, Theorem 3.21 in the book, and Exercises 1 – 3 in (3.2), and 1, 2 in (3.3).

Outlines of (3.4), (3.5): A question concerned here is:

Q: When can a mapping (or a transformation) \( f : \mathbb{R}^n \to \mathbb{R}^n \) of class \( C^1 \) have a local inverse? How smooth is the inverse (if it exists)?

Read e.g. the following The Inverse Function Theorem in the book.
Theorem 3.18: Let \( U \) and \( V \) be open sets in \( \mathbb{R}^n, a \in U \), and \( b = f(a) \). Suppose that \( f : U \to V \) is a mapping of class \( C^1 \) and the Fréchet derivative \( Df(a) \) is invertible (i.e. the Jacobian det \( Df(a) \) is nonzero). Then \( \exists \) neighborhoods \( M \subset U \) and \( N \subset V \) of \( a \) and \( b \), respectively, so that \( f \) is a one-to-one map from \( M \) onto \( N \), and the inverse map \( f^{-1} \) from \( N \) to \( M \) is also \( C^1 \). Moreover, if \( y = f(x) \in N, D(f^{-1})(y) = [Df(x)]^{-1} \).

Note that if \( f \) is \( C^1 \) with nonzero det \( Df(a) \) at an interior point \( a \), then \( b = f(a) \) is also an interior point of \( f(U) \), i.e. The Inverse Function Theorem also tells us that when an interior point in the domain is mapped to an interior point in the range.

Also, read e.g. Theorem 3.20, Theorem 3.22 in the book, and Exercises 1, 2, 5 in (3.4) and 1 in (3.5).