Advanced Calculus (II)

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12.4: Change-of-Variables

(1) \[ \int_{\phi(E)} f(u) du = \int_E f(\phi(x)) |\Delta \phi(x)| dx. \]

(2) When \( f = 1 \),

\[ |R| = \int_{\phi^{-1}(R)} |\Delta \phi(x)| dx. \]
Lemma (12.43)

Let $W$ be open in $\mathbb{R}^n$, let $\phi : W \rightarrow \mathbb{R}^n$ be 1-1 and continuously differentiable on $W$ with $\Delta \phi \neq 0$ on $W$, and suppose that $\phi^{-1}$ is continuously differentiable on $\phi(W)$ with $\Delta \phi^{-1} \neq 0$ on $\phi(W)$. Suppose further that (2) holds for every $n$-dimensional rectangle $R \subset \phi(W)$. If $E$ is a Jordan region with $\overline{E} \subset W$, if $f$ is integrable on $\phi(E)$, and if $f \circ \phi$ is integrable on $E$, then

$$
\int_{\phi(E)} f(u) \, du = \int_{E} (f \circ \phi)(x) |\Delta \phi(x)| \, dx.
$$
Lemma (12.44)

Let $V$ be open in $\mathbb{R}^n$ and $\phi : V \rightarrow \mathbb{R}^n$ be 1-1 and continuously differentiable on $V$. If $\Delta \phi (a) \neq 0$ for some $a \in V$, then there exists an open rectangle $W$ such that $a \in W \subset V$, $\Delta \phi$ is nonzero on $W$, $\phi^{-1}$ is $C^1$ and its Jacobian is nonzero on $\phi(W)$, and such that if $R$ is an $n$-dimensional rectangle contained in $\phi(W)$, then $\phi^{-1}(R)$ is Jordan and (2) holds.
Lemma (12.45)

Suppose that $V$ is open in $\mathbb{R}^n$, $a \in V$, and $\phi : V \rightarrow \mathbb{R}^n$ is continuously differentiable on $V$. If $\Delta \phi (a) \neq 0$, then there exists an open rectangle $W \subset V$ containing $a$ such that if $E$ is Jordan with $\overline{E} \subset W$, if $f \circ \phi$ is integrable on $E$, and if $f$ is integrable on $\phi(E)$, then

\[
(37) \quad \int_{\phi(E)} f(u) \, du = \int_{E} f(\phi(x)) |\Delta \phi(x)| \, dx.
\]
Theorem (12.46)

Suppose that $V$ is open in $\mathbb{R}^n$, and that $\phi : V \to \mathbb{R}^n$ is 1-1 and continuously differentiable on $V$. If $\Delta \phi \neq 0$ on $V$, if $E$ is Jordan region with $\overline{E} \subset V$, if $f \circ \phi$ is integrable on $E$, and if $f$ is integrable on $\phi(E)$, then

$$\int_{\phi(E)} f(u) \, du = \int_E f(\phi(x)) |\Delta \phi(x)| \, dx.$$
Example (spherical coordinates)

\[ x = \rho \sin \varphi \cos \theta, \]
\[ y = \rho \sin \varphi \sin \theta, \]
\[ z = \rho \cos \varphi. \]
Example (12.50)

Find

\[ \int \int \int_{Q} x \, dV, \]

where \( Q = B_3(0,0,0) \setminus B_2(0,0,0) \).
Thank you.