Advanced Calculus (II)

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9.3: Continuous functions

Definition (9.22)

Let $E$ be a nonempty subset of $\mathbb{R}^n$ and let $f : E \to \mathbb{R}^m$. 
(i) $f$ is said to be continuous at $a \in E$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ (which in general depends on $\varepsilon$, $f$, and $a$) such that

$$(3) \quad \|x - a\| < \delta \text{ and } x \in E \quad \text{imply} \quad \|f(x) - f(a)\| < \varepsilon.$$ 

(ii) $f$ is said to be continuous on $E$ (notation: $f : E \to \mathbb{R}^m$ is continuous) if and only if $f$ is continuous at every $x \in E$. 

Definition (9.23)

Let $E$ be a nonempty subset of $\mathbb{R}^n$ and let $f : E \to \mathbb{R}^m$. Then $f$ is said to be \textit{uniformly continuous} on $E$ (notation: $f : E \to \mathbb{R}^m$ is uniformly continuous) if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|x - a\| < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad \|f(x) - f(a)\| < \varepsilon.$$
Theorem (9.24)

Let $E$ be a nonempty compact subset of $\mathbb{R}^n$. If $f$ is continuous on $E$, then $f$ is uniformly continuous on $E$. 
Proof.

Suppose that $f$ is continuous on $E$. Given $\varepsilon > 0$ and $a \in E$, choose $\delta(a) > 0$ such that

$$x \in B_{\delta(a)}(a) \text{ and } x \in E \text{ imply } \|f(x) - f(a)\| < \frac{\varepsilon}{2}.$$ 

Since $\delta(a)/2$ is positive for all $a \in E$, we can choose finitely many points $a_j \in E$ and numbers $\delta_j := \delta(a_j)/2$ such that

$$(4) \quad E \subset \bigcup_{j=1}^{N} B_{\delta_j}(a_j).$$

Set $\delta := min\{\delta_1, \ldots, \delta_N\}$. 


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Set $\delta := \min\{\delta_1, \ldots, \delta_N\}$. 

\[\square\]
Proof.

Suppose that \( x, a \in E \) and \( \|x - a\| < \delta \). By (4), \( x \) belongs to \( B_{\delta_j}(a_j) \) for some \( 1 \leq j \leq N \). Hence,

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\|a - a_j\| \leq \|a - x\| + \|x - a_j\| < \delta_j + \delta_j = 2\delta_j = \delta(a_j),
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i.e., \( a \) also belongs to \( B_{\delta(a_j)}(a_j) \). It follows, therefore, from the choice of \( \delta(a_j) \) that

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\|f(x) - f(a)\| \leq \|f(x) - f(a_j)\| + \|f(a_j) - f(a)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
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This proves that \( f \) is uniformly continuous on \( E \).
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\qed
Theorem (9.25)

Let $n, m \in \mathbb{N}$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then the following three conditions are equivalent.

(i) $f$ is continuous on $\mathbb{R}^n$.

(ii) $f^{-1}(V)$ is open in $\mathbb{R}^n$ for every open subset $V$ of $\mathbb{R}^m$.

(iii) $f^{-1}(E)$ is closed in $\mathbb{R}^n$ for every closed subset $E$ of $\mathbb{R}^m$. 

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Advanced Calculus (II)
Theorem (9.26)

Let $n, m \in \mathbb{N}$, let $E$ be open in $\mathbb{R}^n$, and suppose that $f : E \to \mathbb{R}^m$. Then $f$ is continuous on $E$ if and only if $f^{-1}(V)$ is open in $E$ for every open set $V$ in $\mathbb{R}^m$. 
Example (9.27)

(i) If \( f(x) = \frac{1}{x^2+1} \) and \( E = (0, 1] \), then \( f \) is continuous on \( \mathbb{R} \) and \( E \) is bounded, but \( f^{-1}(E) = (-\infty, \infty) \) is not bounded.

(ii) If \( f(x) = x^2 \) and \( E = (1, 4) \), then \( f \) is continuous on \( \mathbb{R} \) and \( E \) is connected, but \( f^{-1}(E) = (-2, -1) \cup (1, 2) \) is not connected.
Theorem (9.29)

Let $n, m \in \mathbb{N}$. If $H$ is compact in $\mathbb{R}^n$ and $f : H \to \mathbb{R}^m$ is continuous on $H$, then $f(H)$ is compact in $\mathbb{R}^m$. 
Theorem (9.30)

Let \( n, m \in \mathbb{N} \). If \( E \) is connected in \( \mathbb{R}^n \) and \( f : E \to \mathbb{R}^m \) is continuous on \( E \), then \( f(E) \) is connected in \( \mathbb{R}^m \).
Remark (9.31)

The graph $y = f(x)$ of a continuous real function $f$ on an interval $[a, b]$ is compact and connected.
Theorem (9.32 Extreme Value Theorem)

Suppose that $H$ is a nonempty subset of $\mathbb{R}^n$ and $f : H \to \mathbb{R}$. If $H$ is compact, and $f$ is continuous on $H$, then

$$M := \sup\{f(x) : x \in H\} \quad \text{and} \quad m := \inf\{f(x) : x \in H\}$$

are finite real numbers. Moreover, there exist points $x_M, x_m \in H$ such that $M = f(x_M)$ and $m = f(x_m)$. 
Theorem (9.33)

Let $n, m \in \mathbb{N}$. If $H$ is a compact subset of $\mathbb{R}^n$ and $f : H \to \mathbb{R}^m$ is 1-1 and continuous, then $f^{-1}$ is continuous on $f(H)$. 
Thank you.