Part I. Analysis

(1) Let $f : [0,1] \to \mathbb{R}$ be differentiable and such that there is no x for which $f(x) = f'(x) = 0$. Show that the set $Z = \{x \in [0,1] : f(x) = 0\}$ is finite. 10%

(2) Let f and g be continuous real-valued funtions on $[a,b]$. Assume that f is nondecreasing and g and $1-g$ are strictly positive. Put 10%

$$\varphi(x) = \int_a^x f(t)g(t)\,\mathrm{d}t \quad \text{and} \quad \psi(x) = \int_a^{a+G(x)} f(t)\,\mathrm{d}t,$$

where $G(x) = \int_a^x g(t)\,\mathrm{d}t$. Which is larger, $\varphi(x)$ or $\psi(x)$?

(3) Let m be the Lebesque measure on $[0,1]$ and define $\| f \|_p$ with respect to m. Find all functions Φ on $[0,\infty)$ such that the relation 15%

$$\Phi\Big(\lim_{p\to\infty} \| f \|_p\Big) = \int_0^1 (\Phi \circ f)\,\mathrm{d}m$$

holds for every bounded measurable non-negative real-valued f.

(4) Let X be a real Banach space with topological dual X^* and $L \in X^* \setminus \{0\}$ and $H := \{y \in X : L(y) = c\}$ where $c \in \mathbb{R}$ is a constant. For $x \in X$, put $\mathrm{d}(x,H) = \inf\{\| x - y \| : y \in H\}$. Show that $\mathrm{d}(x,H) = \dfrac{|L(x) - c|}{\| L \|}$. 15%

Part II. Algebra

Notation. In the following problem set, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ deonte the sets of natural numbers, integers, rational numbers and real numbers, respectively. The set $\mathbb{Z}_m$ is the ring of integers modulo m ($m > 0$). A *principal ideal ring* is a ring with the property that any ideal is generated by a single element. For a homomorphism f from an algebraic structure to another algebraic structure, $\ker f$ is the kernel of f and $Im\, f$ is the image of f, respectively. A *skew-field* is a (possibly) noncommutative field (i.e., $a \cdot b$ may not be the same as $b \cdot a$). An element a in an algebra A is a *zero-divisor* if there exists some $b \in A$ such that $a \cdot b = 0$.

(5) (i) Show that $\mathbb{Z}$ is a principal ideal ring. 10%

(ii) Show that every homomorphic image of a principal ideal ring is also a principal ideal ring.

(iii) Show that $\mathbb{Z}_m$ is a principal ring for every $m \in \mathbb{N}$.

(6) Let R be a ring, A an R-module and $f : A \to A$ is an R-module homorphism such that $f^2 = f$. Show that $A = \ker f \oplus Im\, f$. 10%

(7) Let A be a finite dimensional algebra over a field F. If A contains no nonzero zero-divisors, then A is a skew-field. 10%

(8) Show that every $m \times n$ matrix over $\mathbb{R}$ of rank 1 is a product of an $m \times 1$ matrix and a $1 \times n$ matrix. 10%

(9) Find all field automorphisms of $\mathbb{Q}$. 10%

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