Part I

1. Let A be a 2×3 matrix. Show that there exist an invertible 2×2 matrix P and an invertible 3×3 matrix Q such that PAQ is of the form $\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$. (10%)

2. Assume that T is a linear operator on a complex inner product space V. Show that (10%)
   (i) If $\langle Tx, x \rangle = 0$, for all $x \in X$, then $T = 0$.
   (ii) If $\langle Tx, x \rangle \in \mathbb{R}$, for all $x \in X$, then $T = T^*$.

3. Let V be a finite dimensional vector space and V^* the dual space of V. Prove that if W is a subspace of V, then $\dim(W) + \dim(W^\circ) = \dim(V)$, where $W^\circ = \{f \in V^* : f(x) = 0 \text{ for all } x \in W\}$. (10%)

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the characteristic polynomial $f(t) = p(t)q(t)$. Suppose $p(t)$ and $q(t)$ are relative prime. (10%)
   (i) Prove that $N(p(T)) \subseteq R(q(T))$ where $N(p(T))$ and $R(q(T))$ denote the null space of $p(T)$ and the range of $q(T)$, respectively.
   (ii) Does $N(p(T)) = R(q(T))$? Justify your answer !

5. Suppose A, B and D are square matrices such that $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$. Show that $\det A = (\det B)(\det D)$. (10%)

Part II

1. (i) Suppose f and g are (Riemann) integrable on $[a, b]$ with $g(x) \geq 0$ for all $x \in [a, b]$. Please write down the mean value theorem for integrals. (10%)
   (ii) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and satisfies $\int_0^1 f(x)dx = \frac{1}{2}$. Does f have a fixed point ξ in $[0, 1]$? (10%)

2. Let (X, d) be a metric space. If T maps X into X and if there is $c \in (0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$, then T is said to be a contraction of X into X. The Banach contraction principle says that if T is a contraction of a complete metric space X into itself, then T has a unique fixed point ξ in X. Please use this principle to answer the following problem: Show that there is a unique continuous function $f : [-1, 1] \to \mathbb{R}$ such that $f(x) = x + \frac{1}{2}\sin f(x)$. (10%)

3. Suppose $\alpha > 1$. Evaluate the limit $\lim_{n \to \infty} \int_0^1 \frac{x \sin x}{1 + (nx)^\alpha} dx$. (10%)

4. Prove that there exist functions $u(x, y)$, $v(x, y)$ and $w(x, y)$ and an $r > 0$ such that u, v, w are C^1 and satisfy the equations

$$u^5 + xv^2 - y + w = 0$$

$$v^5 + yu^2 - x + w = 0$$

$$w^4 + y^5 - x^4 = 1$$

on the ball $B((1, 1); r)$ of $\mathbb{R}^2$, and $u(1, 1) = 1$, $v(1, 1) = 1$, $w(1, 1) = -1$. (10%)