PART I

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, define

$$\phi_g(f) = \int fg\, d\mu.$$

Show that the mapping $g \mapsto \phi_g$ is not injective (one–to–one) from $L^\infty(\mu)$ to $(L^1(\mu))^*$ if μ is not semifinite. Here $(L^1(\mu))^*$ is the dual space of $L^1(\mu)$.

2. Let $\mathcal{H}$ be a Hilbert space, $\{u_\alpha\}_{\alpha\in A}$ an orthonormal set in $\mathcal{H}$, and $\mathcal{M} = \operatorname{span}\{u_\alpha\}_{\alpha\in A}$. Show that for all $x \in \mathcal{H}$,

$$x - \sum_{\alpha\in A} (x, u_\alpha) u_\alpha$$

is perpendicular to $\bar{\mathcal{M}}$, the closure of $\mathcal{M}$.

3. Let $E \subset \mathbf{R}$ be a Lebesgue measurable set with $m(E) > 0$. Show that for any $\alpha < 1$ there exists an open interval I such that $m(E \cap I) > \alpha\, m(I)$. Here m is the Lebesgue measure.

4. Let X be a locally compact Hausforf space. Show that if $\{U_n\}_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X.

5. Show that if $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

Part II.

6. Let S be a nonempty set of automorphisms of a field F. S is linearly independent provided that for any $a_1, \ldots, a_n \in F$ and $\sigma_1, \ldots, \sigma_n \in S$ $(n \geq 1)$:

$$a_1\sigma_1(u) + \cdots + a_n\sigma_n(u) = 0 \quad \text{for all } u \in F \Rightarrow a_i = 0 \quad \text{for every } i.$$

Prove that if S is a set of distinct automorphisms of a field F, then S is linearly indpendent.

7. Let K be a commutative ring with identity. If A is an $n \times m$ matrix over K and B an $m \times n$ matrix over K, then $x^m P_{AB} = x^n P_{BA}$ where P_{AB}, P_{BA} are characteristic polynomials of the matrices AB, BA respectively. Furthermore, if $m = n$, then $P_{AB} = P_{BA}$.

8. Determine the structure of the abelian group G defined by generators a, b, c and relations $3a + 9b + 9c = 0$ and $9a - 3b + 9c = 0$.

9. An element e in a ring R is said to be idempotent if $e^2 = e$. An element of the center of the ring R is said to be central. If e is a central idempotent in a ring R with identity, then

   (a) $1_R - e$ is a central idempotent;

   (b) eR and $(1_R - e)R$ are ideals in R such that

$$R = eR \times (1_R - e)R.$$

10. A ring R is called a Boolean ring if $a^2 = a$ for every element a of R. If R is a Boolean ring and $a \in R$, prove that $2a = 0$. Then prove that R is necessarily a commutative ring.