國立成功大學 110 學年度「碩士班」研究生甄試入學考試

線性代數

1. Find the inverse of

$$\begin{pmatrix} 2 & 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 & 8 \\ 0 & 0 & 9 & 0 & 0 \\ 3 & 0 & 0 & 5 & 0 \\ 0 & 4 & 0 & 0 & 17 \end{pmatrix}$$

(15 points)

2. Show that for any $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$. (15 points)

3. Let $\mathcal{P}_2$ be the real vector space of real quadratic polynomials (polynomials of degree at most 2). Find an orthonormal basis for $\mathcal{P}_2$ with respect to the inner product $\langle f, g\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$. (You do not need to show that it is truely an inner product.) (15 points)

4. For real t show that

$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

(15 points)

5. (a) Find the matrix $P \in \mathbb{R}^{3\times 3}$ such that $x \mapsto Px$ is the orthogonal projection of $\mathbb{R}^3$ onto span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. (10 points)

   (b) Find $\min_{x\in\mathbb{R}^2} \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} x \right\|$. (10 points)

6. (a) Let $A \in \mathbb{C}^{n\times n}$. Prove that if $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$, then A is Hermitian. (10 points)

   (b) Let $A \in \mathbb{R}^{n\times n}$. Is it true that $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$ implies A is symmetric? (10 points)

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高等微積分

1. (10%) Show that $f(x) = \sqrt[3]{x}$ is uniformly continuous on $\mathbb{R}$.

   Hint: find a constant $c > 0$ such that $|a-b|^3 \leq c|a^3 - b^3|$ for all $a, b \in \mathbb{R}$.

2. (15%) Denote $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $\|\mathbf{x}\| = \sqrt{x^2 + y^2}$.

   Let $G \subseteq \mathbb{R}^2$ be an open set such that the closed disk $\{\|\mathbf{x}\| \leq 1\} \subseteq G$.

   Show that there exists $\epsilon > 0$ such that $\{\|\mathbf{x}\| \leq 1 + \epsilon\} \subseteq G$.

3. (15%) Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $x_n \longrightarrow x$. Show that

$$\frac{(2n-1)x_1 + (2n-3)x_2 + \cdots + 3x_{n-1} + x_n}{n^2} \longrightarrow x.$$

4. (15%) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Show that

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

   Also provide an example such that $\limsup\limits_{n\to\infty} \sqrt[n]{x_n} < 1 < \limsup\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$.

5. (15%) Let $f(x)$ be a Riemann integrable function on $[0, 1]$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)\cos(nx)dx = 0.$$

   Hint: first let $f(x)$ be a piecewise constant function.

6. (15%) Determine the values of $p \geq 0$ for which the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{(1+n^2)^p}$$

   is divergent, conditionally convergent or absolutely convergent.

7. (15%) Evaluate the limit. Justify your calculation.

$$\lim_{n\to\infty} \int_0^1 n \cdot \sin\left(\frac{x^2}{n}\right) dx$$