Show all works!

Part I.

1. Let $T : P_2(R) \to P_3(R)$, where $P_n(R)$ consists of all polynomials over real numbers R having degree less than or equal to n; $T(f(x)) = xf(x) + f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$. Find bases for $N(T)$ (the null space of T) and $R(T)$ (the range of T). (10%)

2. Let $V = R^3$, and define $f_1, f_2, f_3 \in V^*$ (the dual space of V) as follows: $f_1(x, y, z) = x - 2y$, $f_2(x, y, z) = x + y + z$, $f_3(x, y, z) = y - 3z$. Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual. (10%)

3. $T : R^3 \to R^3$ defined by

$$T(a_1, a_2, a_3) = (a_1 + 2a_2 + a_3, -a_1 + a_2 + 2a_3, a_1 + a_3).$$

Show that T is invertible and compute the inverse of T. (10%)

4. Let $W = \{(x, y, z) | x + 3y - 2z = 0\}$ be a subspace of the inner product space $V = R^3$ and $y = (3, 2, 1)$. Find the orthogonal projection of y on W. (10%)

5. Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Find a Jordan canonical form J and an invertible matrix Q such that $J = Q^{-1}AQ$. (10%)

Part II.

6. Prove or disprove the following statements: (24%)

(i) If a function f is uniformly continuous on (a, b), then it is differentiable on (a, b);

(ii) If each f_n is continuous on (a, b) and $f_n \to f$ uniformly, then f is continuous on (a, b);

(iii) If the real series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges;

(iv) If f is continuous on $[a, b]$ and $\int_a^b fg = 0$ for every integrable function g on $[a, b]$, then $f(x) = 0$ for all $x \in [a, b]$;

(v) The function $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$ is differentiable on $\mathbb{R}^2$;

(vi) If V is a nonempty, open set in $\mathbb{R}^2$ and $f : V \to \mathbb{R}$ is differentiable on V such that $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ on V, then f is constant on V.

7. Suppose $\{x_n\}$ and $\{y_n\}$ are real sequences. Prove that if $\lim_{n\to\infty} x_n$ exists, then (6%)

$$\lim_{n\to\infty} \sup(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} \sup y_n.$$

8. Suppose f is differentiable on $[a - h, a + h]$ $(h > 0)$. Prove that there exists $\theta \in (0, 1)$ such that (6%)

$$\frac{f(a + h) - f(a - h)}{h} = f'(a + \theta h) + f'(a - \theta h).$$

9. Prove that for all $x \in [0, 1]$ (7%)

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \le \ln(1 + x) \le x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{16}.$$

10 Find $\iint_E e^{\frac{y-x}{y+x}}\, dA$ where E is the region bounded by the lines $x + y = 1$, $x + y = 2$, $x = y$, $x = 0$. (7%)