Show all works

Part I.

1. One way to solve the system of simultaneous equations $\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases}$ is to solve its corresponding matrix equation, $A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, which tells us how the matrix A acts on the vector $\mathbf{x}$. Use this fact to show that

$$AB \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}.$$ (10%)

2. Does the iteration $\mathbf{x}_{k+1} = (I - A)\mathbf{x}_k + \mathbf{b}$ converge for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$?
   State your reason. (10%)

3. If $A + iB$ is a unitary matrix (A and B are real matrices) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is an orthogonal matrix. If $A + iB$ is a Hermitian matrix (A and B are real matrices) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is a symmetric matrix. (10%)

4. Let $A^2 = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$. Find A. (Use the fact that the dimension of null space of A + the rank of $A = 3$.) (10%)

5. Show that, in R^3, the rotation around the unit vector $\mathbf{a} = (a_1\ a_2\ a_3)$ by θ is

$$Q = \cos\theta I + (1-\cos\theta) \begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}.$$ (10%)

Part II.

6. A set S in $\mathbf{R}^n$ is called *convex* if, for every pair of points x and y in S and every real number $t \in (0,1)$, we have $tx + (1-t)y \in S$.
   (1) Prove that the unit closed ball $\{ x \in \mathbf{R}^n \mid \|x\| \le 1 \}$ is convex. (5%)
   (2) Prove that the closure of a convex set is convex. (5%)

7. (1) Let $f_n(x) = x^n$. Show that the sequence $\{f_n\}$ converges pointwise but not uniformly on the interval $[0,1]$. (5%)
   (2) Let g be a continuous function on $[0,1]$ with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0,1]$. (5%)

8. Let $f: \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

   Show that $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ exist at $(0,0)$ but are not equal. (10%)

9. Suppose that $f, f_1, f_2, f_3 \ldots$ are continuous real-valued functions on $[0,1]$ and that $f_n \to f$ uniformly on $[0,1]$ as $n \to \infty$.
   (1) Prove that each $|f|, |f_1|, |f_2|, \ldots$ is integrable on $[0,1]$. (5%)
   (2) Prove that $\int_0^1 |f_n| \to \int_0^1 |f|$ as $n \to \infty$. (5%)

10. (1) Give an example of a continuous function $f: I \to \mathbf{R}$ such that the graph $\{(x, f(x)) \mid x \in I\}$ is not closed in $\mathbf{R}^2$ where I is the open interval $(0,1)$. (5%)
   (2) Does the improper integral

$$\int_{\mathbf{R}^2} \frac{1}{x^2+y^2}\, dx\, dy$$

   converge? Evaluate the integral if it converges. (5%)

([illegible])

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