Part I.

Advanced Calculus **November 13, 2006**

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Guildlines for the test:

- Print as legibly as possible.
- Show all your work. In each problem, correct answers account for at most 40% of the points. The rest 60% go to a complete justification of your answer.

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1. (a) (5 points) State (Do not prove) the *Inverse Function Theorem.*

   (b) (5 points) Let

$$f(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0\ , & \text{if } x = 0 \end{cases}$$

   Can you use the inverse function theorem to conclude that f is a one-one mapping around a small neighborhood containing 0?

2. (10 points) Let

$$g(x,y) = \begin{cases} 0\ , & \text{if } x > y; \\ 0\ , & \text{if } x = y,\ x \text{ is rational}; \\ 1\ , & \text{if } x = y,\ x \text{ is irrational}; \\ 2y, & \text{if } x < y. \end{cases}$$

   Is f double integral? If "yes", explain why and find the double integral of f on $I = [0,1] \times [0,1]$. If "no", explain why not?

3. (10 points) Let $\triangle \in \mathbb{R}^2$ be the triangular whose vertices are located at $(0,0), (2,0), (1,\sqrt{3})$. Find the double integral

$$\iint_\triangle \frac{dx\,dy}{(1+x^2+y^2)^2}.$$

4. Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

   as (x,y) approaches to (0,0) along the following different curves:

   (a) (5 points) The arc $r = \sin 3\theta,\ \ \frac{\pi}{6} < \theta < \frac{\pi}{3}$.

   (b) (5 points) The differentiable curve $y = f(x)$, given that $f(0) = 0$.

5. (10 points) Let $f_n(x) = \dfrac{1}{nx+1}$ and $g_n(x) = \dfrac{x}{nx+1}$ be defined on [0,1] for $n = 1, 2, 3, \ldots$. Find the limit function of $\{f_n\}$ and $\{g_n\}$ and discuss whether or not the convergence is uniform.

Part II. Linear Algebra.

(6) (8%) Let V be the vector space consisting of all polynomials of degree at most two.

(a) Let $T : V \to V$ be the linear operator $T(f(x)) = f(x+1)$ for any $f(x) \in V$. Find the matrix representation A of T relative to the ordered basis $\{1, x, x^2\}$.

(b) Let $D : V \to V$ be the differential operator $D(f(x)) = f'(x)$ for any $f(x) \in V$. Find the matrix representation B of D relative to the ordered basis $\{1, x, x^2\}$.

(c) Can you find an invertible matrix P, such that $P^{-1}AP = B$?

(7) (8%) Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(a) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(b) Define

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$$

where $A^0 = I$ is the 3×3 identity matrix. Find all eigenvalues of e^A.

(8) (6%) Let A, B be an $n \times n$ matrices satisfying ABA=0. Prove that

$$2\,\text{rank}(A) + \text{rank}(B) \leq 2n.$$

(9) (10%) Prove or disprove the following statements. If you think it is right, give an example.

(a) There is a 3×3 matrix T, $T^4 = 0$ but $T^3 \neq 0$.

(b) There are 3×3 matrices A, B, $\text{tr}(AB) \neq \text{tr}(BA)$.

(10) (6%) Show that every $m \times n$ matrix of rank 1 is a product of an $m \times 1$ matrix and a $1 \times n$ matrix.

(11) (6%) Let V be a finite-dimensional *complex* inner product space, and T be a linear operator on V. Prove that T is seft-adjoint if and only if $< Tx, x >$ is real for every $x \in V$.

(12) (6%) Let A be a *real* skew-symmetric matrix ($A^t = -A$). Show that $A - I$ is invertible.