# Part I. Advance Calculus

**Show all your work** for full credits. Good Luck!

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1. (15 points) Let $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$ be a sequence of functions converging to $f$ **uniformly** in $A$. Prove or disprove the followings
   (a) Assume in addition that $\{f_n\}$ and $f$ are integrable and $A = [0, +\infty)$. Then $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$.
   (b) Assume in addition that $\{f_n\}$ are continuous functions. Then $f$ is continuous.
   (c) Assume in addition that $\{f_n\}$ are differentiable functions. Then $f$ is differentiable and
   $$\lim_{n\to\infty} f_n'(x) = f'(x), \forall x \in A$$
   (Recall: $f_n \to f$ uniformly in $A$ if for any $\varepsilon > 0$, there exists $N$ such that $|f_n(x) - f(x)| < \varepsilon$, for any $n \geq N$ and any $x \in A$.)

2. (12 points) Prove or disprove the followings
   (a) If $f$ is continuous at $x = 0$, then there exists a $\delta > 0$ such that $f$ is continuous on $(-\delta, \delta)$.
   (b) Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Then $f'$ is continuous everywhere.

3. (15 points) Evaluate $\iint_\Omega xy\ dxdy$, where $\Omega$ is the first-quadrant region (i.e. $\{(x, y) : x \geq 0, y \geq 0\}$) bounded by the curves
   $$x^2 + y^2 = 4,\ x^2 + y^2 = 9,\ x^2 - y^2 = 1,\ x^2 - y^2 = 4.$$

4. (a) (10 points) Let $f : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Consider a smooth surface given by $S = \{(x, y, z) : f(x, y, z) = k\}$ for some constant $k$. Show that $\nabla f(\alpha(t))$ is perpendicular to $\alpha'(t)$, for any curve $\alpha(t)$ lying on $S$, provided that $\nabla f(\alpha(t)) \neq 0$.
   (b) (5 points) Find the normal vector of the surface $z = f(x, y) = \sqrt{x^2 + y^2}$ at the point $p = (1, 1, \sqrt{2})$.

5. (15 points) Assume $x_i > 0, 1 \leq i \leq n$. Find the maximum value of the function $f(x) = (x_1 x_2 \cdots x_n)^{1/n}$ subjecting to the constraint that $g(x_1, \cdots, x_n) = x_1 + x_2 + \cdots x_n = n$. (Reminder: explain carefully why your answer is the maximum value but not minimum)

6. (16 points) Let $F$ be a continuous vector field defined in $D \subseteq \mathbb{R}^2$, the unit disk centered at origin. Let $\gamma : [0, 1] \to D$ be a piecewise smooth curve in $D$. Prove that the followings are equivalent:
   (a) $F$ is the gradient of a continuously differentiable function $f$ in $D$.
   (b) $\int_\gamma F \cdot d\gamma = 0$ for every piecewise smooth closed curve $\gamma$. (i.e. $\gamma(0) = \gamma(1)$)

7. (12 points) Let $f : (0, 1) \to \mathbb{R}$ be a continuous function satisfying the condition
   $$f(\frac{x+y}{2}) \leq \frac{f(x) + f(y)}{2},$$
   for any $x, y \in (0, 1)$. Prove that $f$ is convex. (Recall: A function $f$ is convex on $(0, 1)$ if
   $$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y),$$
   for any $0 \leq \lambda, x, y \leq 1$.)

Part II. 

# *Linear Algebra*

1. Let $T$ be a linear operator on $\mathbb{R}^3$ and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ be the matrix representation of $T$ relative to the standard basis. Find the matrix representation of $T$ relative to the ordered basis $\beta = \{(1,1,1),(1,1,0),(1,2,3)\}$. (12%)

2. Suppose $V$ and $W$ are finite-dimensional vector spaces and $T : V \to W$ is linear. Prove that

$$\dim V = \dim(\ker T) + \dim(\operatorname{ran} T).$$

(16%)

3. Let $A, B \in \mathbb{C}^{n\times n}$ and $\lambda$ be an eigenvalue of $A$.
   Show that
   (a) If $B$ is similar to $A$, then $\lambda$ is an eigenvalue of $B$. (8%)
   (b) $p(\lambda)$ is an eigenvalue of $p(A)$ for every polynomial $p(x)$. (12%)

4. (a) Let $V$ be a complex inner product space and $T$ is a linear operator on $V$. Prove that if $\langle Tx, x\rangle = 0$ for all $x \in V$, then $T = 0$. (12%)
   (b) Does the conclusion in (a) hold if $V$ is a real inner product space? Why? (6%)

5. (a) Apply the Gram-Schmidt process to the set $\{(1,1,1),(1,1,0),(1,2,3)\}$ to find an orthonormal basis for $\mathbb{R}^3$. (6%)
   (b) Factor the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ into a product of an orthogonal matrix and an upper triangular matrix. (12%)

6. Prove or disprove the following statements:
   (a) There exists $A \in \mathbb{C}^{n\times n}$ such that

$$\dim\Big(\operatorname{span}\{I, A, A^2, A^3, \ldots\}\Big) = n^2.$$

(8%)

   (b) There exists $A \in \mathbb{C}^{n\times n}$, $A \neq \alpha I$ for any $\alpha \in \mathbb{C}$, such that

$$\dim\Big(\operatorname{span}\{I, A, A^2, A^3, \ldots\}\Big) < n.$$

(8%)