(1) (15%)
  (1a) Please define what is a *compact set*?
  (1b) Use (1a) to verify whether the following sets are compact or not?

$$(0,1), \quad [0,1), \quad (0,1], \quad [0,1].$$

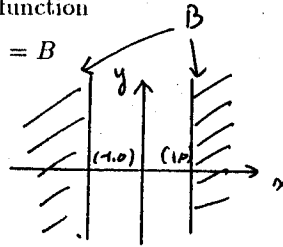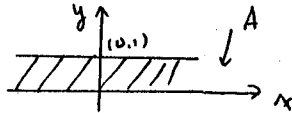 (1c) Please describe the open, connected and compact sets in **R**.

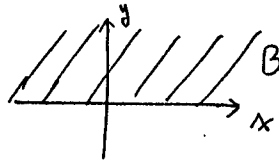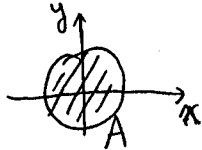(2) (10%) Prove or disprove there exists a continuous function

$$f : \mathbf{R}^2 \mapsto \mathbf{R}^2 \quad \text{such that} \quad f(A) = B$$

where A and B are given by

  (2a)

  (2b)

(3) (10%) Given a set

$$A = \left\{ x^\alpha \sin \frac{1}{x} \,\middle|\, x \in (0, 2\pi] \right\} \qquad \alpha \geq 0$$

Is A a connected set ? What is the colsure of A, i.e., $\bar{A}$?

(4) (10%)
  (4a) Given a function $\{x^\alpha | x \in [0,1]\}, \alpha > 0$. Prove or disprove it is uniform continuous?
  (4b) Given a function $\{x^\alpha | x \in [0,\infty)\}, \alpha > 0$. Prove or disprove it is uniform continuous?

(5) (15%) Given a sequence of trigonometric functions

$$\left\{ \sin^2 nx \,\middle|\, x \in [0, 2\pi] \right\} \qquad n = 1,2,3 \cdots$$

Does it converge uniformly? What is the limit of the integral

$$\int_0^{2\pi} \sin^2 nx\, f(x)\, dx \qquad f \in C[0, 2\pi]$$

as $n \to \infty$.

(6) (20%) Given an improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

Does it converge? Is it absolutely convergent? If converge please compute the integral.

(7) (20%) Prove that the double improper integral

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(ax^2 + 2bxy + cy^2)}\, dx\, dy \qquad a > 0, \quad ac > b^2$$

converges. Could You evaluate the integral? (You must explain every step rigourously.)