

Answer all questions and justify your answers

1. Determine the values of p and q for which the following integrals converge: (24%)

(a) $\int_0^{\frac{\pi}{2}} x^p (\sin x)^q dx$ (b) $\int_1^2 (\ln x)^p dx$

(c) $\int_0^1 x^p (-\ln x)^q dx.$

2. (a) Let $\lim_{n \rightarrow \infty} x_n$ exist and let $y_n = \frac{x_1 + \dots + x_n}{n}$, $n \in \mathbb{N}$.
Prove that $\lim_{n \rightarrow \infty} y_n$ exists. (10%)

(b) Let $\sum_{n=1}^{\infty} a_n$ be convergent and let $c_n = \frac{a_1 + 2a_2 + \dots + na_n}{n(n+1)}$.
Prove that $\sum_{n=1}^{\infty} c_n$ converges and equals $\sum_{n=1}^{\infty} a_n$. (10%)

3. Evaluate the following integrals: (16%)

(a) $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$ (Hint: $\int_0^{\infty} \frac{\sin tx}{x} dx = \frac{\pi}{2}$, $t > 0$)

(b) $\int_0^{\infty} \frac{\cos y}{1+y^2} dy$ (Hint: Set $F(t) = \int_0^{\infty} \frac{\sin tx}{x(x^2+1)} dx$)

4. Let $f : [0, 1] \rightarrow \mathbb{R}$. The n th Bernstein polynomial for f is defined to be

$$B_n(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

(a) Prove that B_n converges uniformly on $[0, 1]$ to f when f is continuous on $[0, 1]$. (10%)

(b) Prove that if $g : [a, b] \rightarrow \mathbb{R}$ is continuous, then g can be uniformly approximated by polynomials. (10%)

5. Every rational x can be written in the form $x = \frac{m}{n}$, where $n > 0$ and m, n are integers without any common divisors, when $x = 0$, take $n = 1$.

Consider function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{n}, & x = \frac{m}{n} \end{cases}$

(a) Prove that f is continuous at every irrational point. (10%)

(b) Prove that f is discontinuous at every rational point. (10%)