

### Entrance Exam, Advanced Calculus, April, 1999

Show all works

1. (a)[10] State the fundamental theorem of calculus.

(b)[10] Evaluate the integral  $\int_0^1 \left( \int_x^1 \frac{\sin y}{y} dy \right) dx$ .

2. (a)[10] Discuss the convergence of the integrals

$$\int_0^1 \frac{1}{x^p} dx \quad \text{and} \quad \int_1^\infty \frac{1}{x^p} dx.$$

(b)[10] Suppose that the series  $\sum_{n=1}^\infty a_n$  converges and for each  $a_n \geq 0$ . Discuss the convergence of the series

$$\sum_{n=1}^\infty \sqrt{a_n} n^{-p}, \quad p \in \mathbb{R},$$

on which interval the series converges and on which interval the series may or may not diverge. If it is in the latter case, please give examples.

3. (a) Suppose that  $\{f_n(x)\}$  and  $\{g_n(x)\}$  converge uniformly on the interval  $[a, b]$ .

(1)[5] Show that the sequence  $\{f_n(x) + g_n(x)\}$  converges uniformly on the interval  $[a, b]$ .

(2)[5] Does  $\{f_n(x)g_n(x)\}$  converge uniformly on the interval  $[a, b]$ ? Prove it or give a counterexample!

(b)[10] Assume that  $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ . Evaluate the integral  $\int_0^\infty \frac{\sin x \cos xy}{x} dx$ .

4. (a)[10] Prove that the surface area of a sphere with radius  $r$  is  $S = 4\pi r^2$ .

(b)[10] Prove that the volume of a torus with radii  $a$  and  $b$  are  $2\pi^2 a^2 b$ .

5. (a)[8] Please state the definitions of an open set, a closed set, a connected set, and a compact set in a metric space.

(b)[6] For the following statement, state your reason why you think it is right or wrong.

There is a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $f(A) = B$  where  $A = \{(x, y) | 0 < y < 1\}$  and  $B = \{(x, y) | x > 1 \text{ or } x < -1\}$ .

(c)[6] For the following statement, state your reason why you think it is right or wrong.

There is a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $f(A) = B$  where  $A = \{(x, y) | x^2 + y^2 \leq 1\}$ .