1. [10%] Find the extreme points (x, y, z) of the function $f(x, y, z) = x + y + z$ subject to the conditions $x^2 + y^2 = 2$ and $x + z = 1$.

2. [8%] Evaluate the definite integral:

$$\iint_A e^{-x^2}\, dx\, dy$$

where A is the triangle on the xy-plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

3. Let $f: \mathbf{R}^n \to \mathbf{R}$ be a function such that the closure of the set $\{ x \mid f(x) \neq 0 \}$ is compact. Define

$$O(f, x_0) = \inf\{ \sup\{ |f(x_1) - f(x_2)| \mid x_1, x_2 \in U \} \mid U \text{ is a neighborhood of } x_0 \}$$

where x_0 is a point in $\mathbf{R}^n$ and "inf" is taken over all neighborhoods U of x_0.

(i) [4%] Define $g: \mathbf{R} \to \mathbf{R}$ by $g(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $g(0) = 0$. Find $O(g, 0)$.

(ii) [4%] Show that if f is continuous at x_0, then $O(f, x_0) = 0$.

(iii) [8%] Define $D_\epsilon = \{ x \in \mathbf{R}^n \mid O(f, x) \geq \epsilon \}$ for $\epsilon > 0$. Show that D_ϵ is a compact set in $\mathbf{R}^n$.

4. [10%] Define $r_k(x) = \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}$. Show that $\sum_{k=0}^{n} r_k(x) = 1$ and $\sum_{k=0}^{n} k r_k(x) = nx$.

5. [10%] A function $f: [a, b] \to \mathbf{R}$ is said to be *integrable* if for any $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$ where $U(f, P)$ denotes the upper sum and $L(f, P)$ denotes the lower sum for P. Show that if f is continuous on $[a, b]$, then f is integrable. (Note: You are not allowed to apply Lebesgue's theorem.)

6. Define $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ for $x \in \mathbf{R}$.

(i) [6%] Show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in \mathbf{R}$ i.e., $f_n(x)$ converges at every $x \in \mathbf{R}$.

(ii) [6%] Let $\exp(x) = \lim_{n\to\infty} f_n(x)$ for $x \in \mathbf{R}$. Show that the convergence $f_n(x) \to \exp(x)$ is uniformly on the open interval $(-r, r)$ for any $r > 0$. Does $f_n(x)$ converge to $\exp(x)$ uniformly on $(-\infty, \infty)$? Why or why not?

(iii) [8%] Show that $\frac{d}{dx}\exp(x) = \exp(x)$.

7. A map $f: \mathbf{R}^n \to \mathbf{R}^m$ is said to be *differentiable* at a point $x_0 \in \mathbf{R}^n$ if there is a linear map $\mathbf{D}f(x_0): \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - \mathbf{D}f(x_0)(x - x_0)\|}{\|x - x_0\|} = 0.$$

(i) [4%] Suppose that $g: \mathbf{R} \to \mathbf{R}$ is given by $g(x) = x^2$. Find $\mathbf{D}g(2)$.

(ii) [4%] Suppose that $\|f(x)\| \leq M\|x\|^2$ for all $x \in \mathbf{R}^n$ where M is a constant. Show that f is differentiable at $x_0 = 0$ and $\mathbf{D}f(0) = 0$.

(iii) [6%] Suppose that both f, g are differentiable at x_0. Show that $f + g$ is differentiable at x_0 and

$$\mathbf{D}(f + g)(x_0) = \mathbf{D}f(x_0) + \mathbf{D}g(x_0).$$

8. A brief explanation is required for this problem.

(i) [4%] Give an example of a continuous function f and a closed set A in the domain of f such that $f(A)$ is not closed.

(ii) [4%] Give an example of a real-valued differentiable function f which is uniformly continuous on its domain but its derivative is unbounded.

(ii) [4%] Give an example of non-empty closed sets I_k in $\mathbf{R}$ for $k = 1, 2, 3, \ldots$ such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ and $\bigcap_{k=1}^{\infty} I_k = \emptyset$.