1. (15 pts.) Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers. Show that

$$\overline{\lim_{n\to+\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to+\infty}} \frac{a_{n+1}}{a_n}.$$

2. Suppose $f : [a,b] \to \mathbb{R}$ is a C^1 injection.
   (a).(7 pts.) Show that $\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a)$.
   (b).(7 pts.) If $f(x) \ge 0, \forall x \in [a,b]$, give a geometric interpretation for the formula in (a).
   (c).(6 pts.) Evaluate $\int_0^1 \left((x-1)^{\frac{1}{3}} + 1\right)^{\frac{1}{2}} dx$.

3. (15 pts.) Suppose E is a nonempty compact subset of $\mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}$ are C^1 such that $f = g$ on the boundary of E. Show that there is a point $\mathbf{x_0} \in$ E such that $\nabla f(\mathbf{x_0}) = \nabla g(\mathbf{x_0})$.

4. (15 pts.) If $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on every closed subinterval of $(0,1)$, does it follow that $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on $(0,1)$? Support your statement with either a proof or a counterexample.

5. (a).(8 pts.) State the Implicit Function Theorem.
   (b).(7 pts.) Decide whether it is possible to solve the pair of equations

$$xy^2 + xzu + yv^2 - 3 = 0$$
$$u^3yz + 2xv - u^2v^2 - 2 = 0$$

   for u and v as C^1 functions of (x,y,z) in a neighborhood of the points $(u,v) = (1,1)$ and $(x,y,z) = (1,1,1)$.

6. For any $n \in \mathbb{N}$, let $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$.
   (a).(10 pts.) Show that $\{a_n\}_{n\in\mathbb{N}}$ is convergent to γ for some $\gamma \in \mathbb{R}$.
   (b).(10 pts.) Express $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ as $1 + \frac{1}{2} + \cdots + \frac{1}{n} = \gamma + \ln n + \varepsilon_n$ to evaluate $\sum_{k=1}^{+\infty} \frac{1}{k(2k-1)}$.