1. Recall that for a sequence $\{x_n\}_1^{+\infty}$ in $\mathbb{R}$, $\liminf_{n\to+\infty} x_n$ is defined to be

$$\liminf_{n\to+\infty} x_n = \lim_{n\to+\infty}\left(\inf_{k\geq n} x_k\right)$$

Show that if $\lim_{n\to+\infty} y_n$ exists in $\mathbb{R}$, then for any sequence $\{x_n\}_1^{+\infty}$ in $\mathbb{R}$, one has $\liminf_{n\to+\infty}(x_n+y_n) = \liminf_{n\to+\infty} x_n + \lim_{n\to+\infty} y_n$. 15%

2. Suppose $\{f_n\}_1^{+\infty}$ is a sequence of continuous functions on $[0,+\infty)$. Suppose also there is a function g such that for any $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ on $[0,+\infty)$ and $\int_0^{+\infty} g(x)dx$ converges. Prove that if $\{f_n\}_1^{+\infty} \to f$ uniformly on $[0,L)$ for any $L>0$, then $\lim_{n\to+\infty}\int_0^{+\infty} f_n(x)dx = \int_0^{+\infty} f(x)dx$. 15%

3. (a) Suppose $f:[0,+\infty)\to[0,+\infty)$ is continuous and $\int_0^{+\infty} f(x)dx < +\infty$. Does it follow that $\lim_{x\to+\infty} f(x) = 0$? Support your statement with either a proof or a counterexample. 10%

   (b) If the continuity on f in (a) is replaced by uniform continuity, does it follow that $\lim_{x\to+\infty} f(x) = 0$? Support your statement with either a proof or a counterexample. 10%

4. Let $f(x,y)$ be a real-valued function on $\mathbb{R}^2$ which has continuous second partial derivatives. Show that f is harmonic if and only if

$$\int_C \left(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx\right) = 0$$

for all circles C in the counterclockwise direction in $\mathbb{R}^2$.
(Recall that f is harmonic if and only if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv 0$ on $\mathbb{R}^2$.) 15%

5. Let E be a region in $\mathbb{R}^2$ which is bounded by $3y-x=0$, $3y-x+1=0$, $y+2x=0$, and $y+2x-1=0$. Evaluate

$$\iint_E \sqrt[3]{2x^2-5xy-3y^2}\,d(x,y). \qquad 15\%$$

6. Let $f(x,y) = (x+y, \sin x + \cos y)$, $(x,y)\in\mathbb{R}^2$.

   (a) Show that a differentiable f^{-1} exists on some open set containing $(0,1)$. 10%

   (b) Compute the matrix representation relative to the standard basis on $\mathbb{R}^2$ for the total derivative of f^{-1} at (0,1). 10%

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