(1) Solve the differential equation

(a) (10 points) $\dfrac{dy}{dx} + 2xy = 2xe^{-x^2}$.

(b) (10 points) $(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$.

(2) Consider the following differential equation

$$t^2y'' - ty' = 0. \qquad (1)$$

(a) (5 points) Verify that $y \equiv 0$ is a solution of (1).

(b) (10 points) Find a non-trivial solution of (1) satisfying $y(0) = y'(0) = 0$.

(c) (5 points) Do results in (a) and (b) violate the Uniqueness theorem?

(3) (10 points) Two functions f and g are said to be *linearly independent* on an interval $\alpha < x < \beta$ if the equation $k_1 f(x) + k_2 g(x) = 0$ holds for all x in the interval only if $k_1 = k_2 = 0$. Assume that f and g are differentiable functions and

$$W(f,g)(x_0) = \det \begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \neq 0$$

for some x_0 in $\alpha < x < \beta$. Show that f and g are *linearly independent* on this interval.

(4) Let the functions p and q be continuous on the open interval $\alpha < x < \beta$, and let y_1 and y_2 be two *linearly independent* solutions of the differential equation

$$L[y] = y'' + p(x)y' + q(x)y = 0, \qquad (2)$$

satisfying the condition $W(y_1, y_2)(x) \neq 0$ at every point in $\alpha < x < \beta$. Note that this implies that for any x_0 in the interval and for any vector $V = \begin{pmatrix} c \\ d \end{pmatrix}$, there is a unique vector $U = \begin{pmatrix} a \\ b \end{pmatrix}$ satisfying that $V = \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} U$.

(a)(10 points) Use the Uniqueness theorem to prove that any solution of the equation (2) on the interval $\alpha < x < \beta$ can be expressed uniquely as a linear combination of y_1 and y_2.

(b)(10 points) Suppose that v is a (nonconstant) function satisfying $y_2 = v(x)y_1$. Find a second order differential equation satisfied by v.

(c)(10 points) Solve the differential equation $(x-1)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$, for $x > 1$, by using the fact that $y_1(x) = e^x$ is a solution of the differential equation.

(5) (10 points) Let $X = \xi t e^{\lambda t} + \eta e^{\lambda t}$, where $\xi \neq 0$ and $\eta \neq 0$ are constant vectors in $\mathbb{R}^n$ and $\lambda \in \mathbb{C}$ is a scalar, find conditions on ξ, η, and λ such that X satisfies the system of the equations $\dfrac{dX}{dt} = AX$, where A is a nonzero $n \times n$ matrix of constant entries.

(6) (10 points) Find the general solution of the system of equations

$$\frac{dY}{dx} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} Y.$$