編號：行 50　系所：數學系應用數學　科目：機率論

(1) Let $A_1, \dots, A_n$ be mutually exclusive events whose union is the whole sample space with $P(A_i) > 0$ for each i. Let B be any event with $P(B) > 0$.

(i) Show the Bayes' Rule: 10%

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_i P(B|A_i)P(A_i)}.$$

(ii) The proportion of people in a given community having a certain disease is 0.005. A test is available to diagnose the disease. If a person has the disease, the probability that the test will produce a positive signal is 0.99. If a person does not have the disease, the probability that the test shows a negative signal is also 0.99. In other words, the test correctly classifies 99% of both diseased and non-diseased individuals. If a person tests positive, what is the probability that the person actually has the disease? 10%

(iii) Do you think the test is accurate? why is/isn't it? 10%

(2) Assume that a mobile computer moves within the region A bounded by the x-axis, the line $x = 1$ and the line $y = x$ in such a way that if (X, Y) denotes the position of the computer at a given time, the joint density of X and Y is given by

$$f(x,y) = \begin{cases} 8xy, & \text{if } (x,y) \in A; \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find $Cov(X,Y) = E(XY) - E(X)E(Y)$. 10%

(ii) Assume that the mobile computer moves from a random position (X, Y) vertically to the point $(X, 0)$, then along the x-axis to the origin. Find the mean and variance of the distance traveled. 10%

(3) Let X denote the number of flaws in a one-inch length of copper wire. The probability mass function of X is

$$P(X = x) = \begin{cases} 0.55, & \text{if } x = 0; \\ 0.35, & \text{if } x = 1; \\ 0.05, & \text{if } x = 2; \\ 0.05, & \text{if } x = 3. \end{cases}$$

The expectation of X is 0.6 flaws per one-inch wire.

(i) What is the expected number of flaws if 100 such wires are sampled? 10%

(ii) By *the weak law of large numbers*(WLLN), what can you say about the expected number of flaws and the actual number of flaws in the one hundred wires sampled? 10%

(iii) Let σ^2 be the variance of X and ν_{100} be the total number of flaws in the 100 wires sampled. Use *the Central Limit Theorem* to approximate the probability that the average number of flaws ($\frac{\nu_{100}}{100}$) deviates 0.6 by $\sigma/10$. (The standard normal random variable has the probability 0.16 that its value will fall above 1). 10%

(背面仍有題目,請繼續作答)

(4) Let $b(k;n,p)$ be the probability that n Bernoulli trials result in k successes where p is the probability for a single success. Then, $b(k;n,p) = \binom{n}{k} p^k q^{n-k}$.

(i) Show that　5%

$$f(k) = \frac{b(k;n,p)}{b(k-1;n,p)} = 1 + \frac{(n+1)p - k}{kq}.$$

(ii) Using the above identity to show that the terms $b(k;n,p)$ decrease monotonically for all integers k greater than np. Moreover, $f(k)$ decreases as k increases.　5%

(iii) Let r be an integer greater than np. Show that $b(k;n,p), k \geq r$ decreases faster than the terms of a geometric series with ratio $1 + \frac{np-r}{rq}$. (Hint: Since $f(k)$ is decreasing, you only have to compare the ratio with $f(r+1)$.)　5%

(iv) Let S_n be the number of successes in n trials. Show that,　5%

$$P(S_n \geq r) \leq b(r;n,p) \frac{rq}{r - np}.$$