系所組別： 數學系應用數學
考試科目： 線性代數

※ 考生請注意：本試題 □可 ☑不可 使用計算機

(1) (8%) Let V be a vector space of dimension 9, and let W_1, W_2 and W_3 be distinct subspaces of V such that $\dim(W_1) = \dim(W_2) = \dim(W_3) = 7$. Find the possible dimension of $W_1 \cap W_2 \cap W_3$.

(2) (8%) Let A be a linear transformation on a finite dimensional, real, vector space V and

$$Ax = \lambda x + \mu y, \quad Ay = -\mu x + \lambda y$$

where x, y are nonzero vectors. Prove that $\det((\lambda^2 + \mu^2)I - 2\lambda A + A^2) = 0$.

(3) (10%) Consider the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 2n & 3n & \cdots & n^2 \end{pmatrix}$$

Find the minimal polynomial and characteristic polynomial of A.

(Hint: $\text{rank}(A) = ?$)

(4) Suppose A and B are $n \times n$ nonsingular matrices with the property $AB = BA$.

(a) (6%) Prove that if v is an eigenvector for A, then Bv is also an eigenvector for A.

(b) (8%) Suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Prove that B is diagonalizable.

(5) (10%) Let A be a linear transformation on a finite-dimensional vector space V. Prove that $\dim(\text{Ker}(A + I)) + \dim(\text{Ker}(A - I)) = \dim(V)$ if and only if $A^2 = I$.

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(6) Let $P_n(\mathbb{R})$ be the vector space of all polynomial functions $p(x)$ which has degree n or less (so $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ for some $c_i \in \mathbb{R}$).

(a) (6%) Let $v_i = (x-2)^i$. Prove that $\{v_0, v_1, v_2, \cdots, v_n\}$ is a basis of $P_n(\mathbb{R})$.

(b) (6%) Consider $P_2(\mathbb{R})$ with the inner product (,) defined by

$$(p(x), q(x)) = \int_0^1 p(x)q(x)\,dx$$

Apply the Gram-Schmidt process to the set $\{1, x, x^2\}$ to find an orthonormal basis.

(c) (8%) Let T be the linear transformation of $P_n(\mathbb{R})$ defined by $T(p(x)) = p(x+1) - p(x)$. Is T diagonalizable?

(d) (10%) Let f_i be the linear function defined by $f_i(p(x)) = p(i)$. (For example, $f_3(x^2-2) = 3^2 - 2 = 7$.) Prove that $\{f_0, f_1, f_2, \cdots, f_n\}$ is a basis of the dual space $(P_n(\mathbb{R}))^*$.

(7) (10%) Let V be a real vector space with the inner product (,) and $u, v_1, v_2, \cdots, v_n \in V$. Suppose that $(v_i, u) > 0$ for all i and $(v_i, v_j) \leq 0$ whenever $i \neq j$. Prove that $\{v_1, v_2, \cdots, v_n\}$ is a linear independent set.

(8) (10%) Prove that if A is a linear transformation on a finite-dimensional inner product space V, then there exist unique transformation A^* such that

$$(A^*x, y) = (x, Ay) \quad \forall\, x, y \in V.$$