Smooth Maps

October 1, 2019

Now that we have established the concept of local coordinates of a manifold $M$, we can define smooth maps between manifolds that preserve properties we have learned from advanced calculus.

1 Smooth Maps

Just like smooth functions, the smoothness of a map $F : M \to N$ at a point $p \in M$ is defined to be the smoothness of the coordinate representation of $F$ at $p$ and $F(p)$:

**Definition 1.1.** Given smooth manifolds $N, M$ of dimensions $n, m$ respectively, a map $F : N \to M$ is said to be smooth at $p$, if there are charts $(U, \varphi)$, $(V, \psi)$ at $p$ and $F(p)$ so that $F(U) \subset V$ and

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) (\subset \mathbb{R}^n) \to \psi(V) (\subset \mathbb{R}^m)$$

is a smooth map in the ordinary sense.

It is now clear that compatibilities of charts on $N$ and $M$ make the definition above independent of choice of coordinate:
The following definitions can now be defined on manifolds:

**Definition 1.2.** A map $F : N \to M$ is smooth if it is smooth at every point.

**Definition 1.3.** A map $F : N \to M$ is called a diffeomorphism if it is smooth, bijective, and $F^{-1}$ is smooth.

Constant maps from $M$ to $N$, identity maps from $M$ to itself, and inclusion maps of an open subset $U$ to $M$, should all be smooth. Indeed, their coordinate representations are constant maps in Euclidean spaces. Some basic properties should hold true for smooth maps:

**Theorem 1.4.** Smooth maps are continuous.

**Theorem 1.5.** Composition of smooth maps is smooth.

It is also important to point out that smoothness is a local property.
Proposition 1.6. Let $N$ and $M$ be smooth manifolds and $F : N \to M$ be a map.

- If every $p \in M$ has an open neighborhood $U$ such that $F|_U$ is smooth, then $F$ is smooth.

- If $F$ is smooth, then its restriction to every open subset is smooth.

The proposition immediately implies

Corollary 1.7. Let $M$ and $N$ be smooth manifolds, and $\{U_\alpha\}_\alpha$ be an open cover of $M$. Suppose that for each $\alpha$, there exists smooth functions $F_\alpha : U_\alpha \to N$ so that for all $\alpha, \beta$,

$$F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}.$$

Then, there exists a smooth function $F : M \to N$ so that $F|_{U_\alpha} = F_\alpha$. 

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Let’s study some example of smooth maps and diffeomorphisms.
Diffeomorphisms are basically the "identifiers" of smooth manifolds without concerns of metrics. Though a manifold may have many distinct smooth structures, they may be related to each other via diffeomorphisms. It is a fascinating (and very deep) subject to classify smooth type of a topological manifold, i.e. smooth structures up to diffeomorphisms.

It is not difficult to show the un-surprising fact that dimensions and boundaries are invariants under diffeomorphisms:

**Theorem 1.8.** An $m$-dimensional smooth manifolds can not be diffeomorphic to an $n$-dimensional manifolds unless $n = m$.

**Theorem 1.9.** Suppose $M$ and $N$ are smooth manifolds with a diffeomorphism $F : M \to N$. Then $F(\partial M) = \partial N$ and $F$ restricts to a diffeomorphism from $\text{Int}M$ to $\text{Int}N$.