Dispersive estimates for self-adjoint operators

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State the problem

Let (X, d, μ) be a metric space endowed with a nonnegative Borel measure μ satisfying the doubling condition: there exists a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \tag{1}$$

for all $x \in X$, $r > 0$ and all balls $B(x, r) := \{y \in X : d(x, y) < r\}$.
In this talk we shall also assume that

$$\mu(B(x, r)) \gtrsim r^n \tag{2}$$

for all $x \in X$ and $r > 0$ and for some $n \geq 1$.

Examples of doubling spaces: Euclidean space $\mathbb{R}^n$ with Lebesgue measure; convex domain $\Omega \subset \mathbb{R}^n$ with Lebesgue measure.

State the problem

Let L be a nonnegative self–adjoint operator on $L^2(X)$. Suppose that L satisfies an $L^1 - L^\infty$ dispersive estimate of the form

$$\|e^{itL}\|_{L^1 \to L^\infty} \lesssim |t|^{-a}. \tag{3}$$

This is frequently the case for many important operators, notably the Laplacian $L = -\Delta$ and its potential perturbations.

We note that the dispersive estimate is a useful property in the study of pdes.

State the problem

Question: Is it possible to deduce similar estimates for the more general class of flows $e^{it\phi(L)}$:

$$\|\psi(L)e^{it\phi(L)}\|_{L^1\to L^\infty} \lesssim |t|^{-b} \tag{4}$$

for appropriate functions ψ and ϕ ?

For instance, if we choose $\phi(L) = \sqrt{L}$, we are asking if a dispersive estimate for the wave flow $e^{it\sqrt{L}}$ can be deduced directly from a corresponding estimate for the Schrödinger flow e^{itL}.

Remarks

- The estimate (4) implies the following estimate:

$$\|e^{it\phi(L)}f\|_{L^\infty} \lesssim t^{-c}\|f\|_{\mathcal{X}}$$

  where $\mathcal{X}$ is some function space such as Besov spaces and Sobolev spaces.
- The flows e^{itL^ν} with $\nu \in (0,1]$ have a strong connection with the fractional Schrödinger equation:

$$u_t + iL^\nu u = 0, \quad u(0,\cdot) = f.$$

- In particular, when $\phi(L) = \sqrt{L}$, the estimate (4) implies the following estimate:

$$\Big\|\frac{e^{it\sqrt{L}}}{\sqrt{L}}f\Big\|_{L^\infty} \lesssim t^{-d}\|f\|_{\mathcal{X}}$$

  which tells us the behavior of the solutions to the wave equation

$$u_{tt} + Lu = 0, \qquad u_t(0,\cdot) = f, \qquad u(0,\cdot) = g.$$

Our assumptions

We assume the following conditions on the nonnegative self–adjoint operator L:

(A1) The Schrödinger flow e^{itL} satisfies a dispersive estimate:

$$\|e^{itL}\|_{L^1\to L^\infty} \lesssim |t|^{-n/2}, \quad |t| \in (0, T_0)$$

where $T_0 \in (0, +\infty]$.

(A2) The kernel $p_t(x,y)$ of e^{-tL} admits a Gaussian upper bound: $\exists C, c > 0$ such that for all $x, y \in X$ and $t > 0$,

$$|p_t(x,y)| \leq \frac{C}{\mu(B(x,\sqrt{t}))} \exp\Big(-\frac{d(x,y)^2}{ct}\Big).$$

Our assumptions

Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function. We denote by (H1) and (H2) the following assumptions on ϕ:

(H1) There exists $0 < m_1 \leq 1$ such that

$$\phi'(r) \sim r^{m_1 - 1} \text{ and } |\phi''(r)| \gtrsim r^{m_1 - 2}, \quad r \geq 1.$$

(H2) There exists $m_2 > 0$ such that

$$\phi'(r) \sim r^{m_2 - 1} \text{ and } |\phi''(r)| \gtrsim r^{m_2 - 2}, \quad 0 < r < 1.$$

• Some examples: if $\phi(r) = r^\nu$ with $\nu \in (0, 1)$, then (H1) and (H2) are satisfied with $m_1 = m_2 = \nu$. If $\phi(r) = \sqrt{1 + r^2}$, then (H1) and (H2) are satisfied with $m_1 = 1$ and $m_2 = 2$.

The following formula is important for our result.

- Subordination formula:

Theorem ([BDDM])

Assume ϕ satisfies (H1) and g is a C^∞ function supported in $[1/2, 2]$. Then there exist $c_0 > 1$, and suitable functions $\rho_t(x, \lambda)$ and $a_t(s, \lambda)$ so that

$$g(\lambda^{-1}\sqrt{x})e^{it\phi(x)} = \rho_t(x, \lambda) + \sqrt{t\lambda^{2m_1}}\,\eta(\lambda^{-2}x)\int e^{ixt\lambda^{2m_1-2}s}a_t(s, \lambda)ds \qquad (5)$$

for all $x, t > 0$ and $\lambda \geq 1$, where $\eta \in C^\infty(\mathbb{R})$ is supported in $[1/5, 5]$ and $\eta \equiv 1$ on $[1/4, 4]$.

We have a similar formula for the case $0 < \lambda < 1$.

[BDDM] The Anh Bui, Piero D'Ancona, Xuan Thinh Duong and Detlef Müller, On the flows associated to self-adjoint operators on metric measure spaces, preprint.

Proof.

Let $k \in \mathbb{Z}$ and $t > 0$. For $\lambda \geq 1$ we denote by $\Psi_\lambda(\xi)$ the Fourier transform of $g(\lambda^{-1}\sqrt{x})e^{it\phi(x)}$, i.e.,

$$\begin{aligned}\Psi_\lambda(\xi) &= \int g(\lambda^{-1}\sqrt{x})e^{it\phi(x)}e^{-ix\xi}dx \\ &= \lambda^2 \int g(\sqrt{u})e^{i[t\phi(\lambda^2 u)-\lambda^2 u\xi]}du.\end{aligned} \tag{6}$$

Let $\tau \in C^\infty(\mathbb{R})$ supported in $[2c_0^{-1}, 2c_0]$ with $\tau \equiv 1$ in $[c_0^{-1}, c_0]$ where c_0 will be determined later. Then by the Fourier inversion formula we have

$$\begin{aligned}g(\lambda^{-1}\sqrt{x})e^{it\phi(x)} = \eta(\lambda^{-2}x) &\int \left(1 - \tau\left(\frac{\xi}{t\lambda^{2m_1-2}}\right)\right)\Psi_\lambda(\xi)e^{i\xi x}d\xi \\ &+ \eta(\lambda^{-2}x)\int \tau\left(\frac{\xi}{t\lambda^{2m_1-2}}\right)\Psi_\lambda(\xi)e^{i\xi x}d\xi \\ =: \rho_t(x,\lambda) + A_t(x,\lambda)&\end{aligned}$$

where $\eta \in C^\infty(\mathbb{R})$ is supported in $[1/5, 5]$ and $\eta \equiv 1$ on $[1/4, 4]$. □

Observe that

$$\partial_u[t\phi(\lambda^2 u) - \lambda^2 u\xi] = \lambda^2 t\phi'(\lambda^2 u) - \lambda^2\xi$$

We note that the integrand in the expression for $\rho_t(x, \lambda)$ is supported where either $\xi < c_0^{-1} t\lambda^{2m_1-2}$ or $\xi > c_0 t\lambda^{2m_1-2}$. In this situation, by (H1) we can choose c_0 large enough so that

$$|\partial_u[t\phi(\lambda^2 u) - \lambda^2 u\xi]| \gtrsim (\lambda^2|\xi| + t\lambda^{2m_1})$$

Hence, by integration by parts in (6), we have for these ξ that

$$|\Psi_\lambda(\xi)| \leq C_{k,g,\phi}\lambda^2(\lambda^2|\xi| + t\lambda^{2m_1})^{-k}, \quad \forall k \geq 0, \lambda \geq 1.$$

This implies

$$|\rho_t(x, \lambda)| \leq C_{k,g,\phi}(t\lambda^{2m_1})^{-k}, k \geq 0.$$

We now estimate the term $A_t(x,\lambda)$. By a change of variables, we have

$$\begin{aligned} A_t(x,\lambda) &= t\lambda^{2m_1-2}\eta(\lambda^{-2}x)\int \tau(s)\Psi_\lambda(t\lambda^{2m_1-2}s)e^{ixt\lambda^{2m_1-2}s}ds \\ &= t\lambda^{2m_1}\eta(\lambda^{-2}x)\int \tau(s)e^{ixt\lambda^{2m_1-2}s}\int g(\sqrt{u})e^{i[t\phi(\lambda^2u)-t\lambda^{2m_1}us]}duds \\ &= \sqrt{t\lambda^{2m_1}}\eta(\lambda^{-2}x)\int e^{ixt\lambda^{2m_1-2}s}a_t(s,\lambda)ds \end{aligned}$$

where

$$a_t(s,\lambda) = \sqrt{t\lambda^{2m_1}}\int \tau(s)g(\sqrt{u})e^{i[t\phi(\lambda^2u)-t\lambda^{2m_1}us]}du.$$

It is clear that $\operatorname{supp} a(\cdot,\lambda)\subset[2c_0^{-1},2c_0]$. Moreover, on the support of g we have $1/4<u<4$. In this situation, by (H1) we have

$$\Big|\frac{\partial^2}{\partial u^2}[t\phi(\lambda^2u)-t\lambda^{2m_1}us]\Big| \gtrsim t\lambda^{2m_1}.$$

It then follows that

$$|a_t(x,\lambda)|\lesssim 1.$$

This implies $|a(s,\lambda)|\lesssim 1$ for all $s\in[2c_0^{-1},2c_0]$ and $\lambda\geq 1$. This proves the required estimate.

Tools

Due to the lacking of regularity assumption on the behaviour of the kernels of the semigroup e^{-tL}, the standard Hardy space, Besov spaces ... might not be the appropriate setting for the dispersive estimates for L. An appropriate replacement is using function spaces associated to operators.

• Hardy spaces associated to operators:

Definition

Let $0 < p \le 1$ and $M \in \mathbb{N}$. A function $a(x)$ supported in a ball $B \subset X$ of radius r_B is called a $(p, 2, M, L)$-atom if there exists a function $b \in D(L^M)$ such that

(i) $a = L^M b$;

(ii) $\mathrm{supp} L^k b \subset B$, $k = 0, 1, \ldots, M$;

(iii) $\|L^k b\|_{L^2(X)} \le r_B^{2(M-k)} \mu(B)^{1/2-1/p}$, $k = 0, 1, \ldots, M$.

[HLMMY] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, Mem. Amer. Math. Soc. 214 (2011)

Tools: Hardy spaces associated to operators

Definition (Atomic Hardy spaces for L)

Given $0 < p \le 1$ and $M \in \mathbb{N}$, we say that $f = \sum \lambda_j a_j$ is an atomic $(p, 2, M, L)$-representation if $\{\lambda_j\}_{j=0}^{\infty} \in \ell^p$, each a_j is a $(p, 2, M, L)$-atom, and the sum converges in $L^2(X)$. The space $H^p_{L,at,M}(X)$ is then defined as the completion of

$$\left\{ f \in L^2(X) : f \text{ has an atomic } (p, 2, M, L)\text{-representation} \right\},$$

with the norm given by

$$\|f\|^p_{H^p_{L,at,M}(X)} = \inf \left\{ \sum |\lambda_j|^p : f = \sum \lambda_j a_j \text{ is an atomic } (p, 2, M, L)\text{-representation} \right\}.$$

Example: Let $L = -\Delta + |x|^2$ on $\mathbb{R}^n$. Let $\rho(x) = \min\{1, |x|^{-1}\}$ for $x \in \mathbb{R}^n$. Let $p \in (0,1]$. A function a is called a (p, ∞, ρ)-*atom* associated to the ball $B(x_0, r)$ if

(i) $\operatorname{supp} a \subset B(x_0, r)$;

(ii) $\|a\|_{L^\infty} \leq |B(x_0, r)|^{-1/p}$;

(iii) $\int x^\alpha a(x) dx = 0$ for all $|\alpha| \leq \lfloor n(1/p - 1) \rfloor$ if $r < \rho(x_0)/4$.

The Hardy space $H^p_{at,\rho}(\mathbb{R}^n)$ is then defined as the set of all functions f which can be expressed in the form $f = \sum_j \lambda_j a_j$ where $(\lambda_j)_j \in \ell^p$ and a_j are (p, ∞, ρ)-atoms. Its norm is given by

$$\|f\|^p_{H^p_{at,\rho}(\mathbb{R}^n)} = \inf\Big\{ \sum_j |\lambda_j|^p : f = \sum_j \lambda_j a_j \Big\}$$

where the infimum is taken over all possible atomic decompositions of f. Then we have $H^p(\mathbb{R}^n) \subsetneqq H^p_{at,\rho}(\mathbb{R}^n) \equiv H^p_L(\mathbb{R}^n)$ for all $p \in (0,1]$.

Tools

• Besov spaces associated to operators

Fix a Littlewood–Paley dyadic partition of unity $\Psi = \{\psi_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}$, i.e., $\psi \in S(\mathbb{R})$ such that $\mathrm{supp}\psi \subset [1/2, 2]$ and

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\lambda) = 1 \text{ on } (0, \infty).$$

and define for all $s \in \mathbb{R}$, $1 \le p, q < \infty$ the *Besov space* $B^{s,L}_{p,q}(X)$ as the completion of the set

$$\left\{ f \in L^2(X) : \|f\|_{B^{s,L}_{p,q}} < \infty \right\}$$

for the norm $\|\cdot\|_{B^{s,L}_{p,q}}$ given by

$$\|f\|_{B^{s,L}_{p,q}} := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{js} \|\psi_j(\sqrt{L})f\|_{L^p} \right)^q \right\}^{1/q}.$$

We note that this definition is independent of the choice of Ψ.

Main results

The following are our main results.

Theorem ([BDDM] (High frequency estimate))

Assume L satisfies (A1) and (A2), ϕ satisfies (H1), and $\psi \in C^\infty(\mathbb{R})$ is supported in $[1/2, 2]$. Then we have

$$|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}f| \lesssim |t|^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}\|f\|_{L^1}, \quad \lambda \geq 1,\ |t| < T_0. \tag{7}$$

Theorem ([BDDM] (Low frequency estimate))

Assume L satisfies (A1) and (A2), ϕ satisfies (H2), and $\psi \in C^\infty(\mathbb{R})$ is supported in $[1/2, 2]$. Then we have

$$|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}f| \lesssim |t|^{-\frac{n-1}{2}}\lambda^{(1-m_2)n+m_2}\|f\|_{L^1}, \quad 0 < \lambda < 1,\ |t| < T_0. \tag{8}$$

We give the proof for the case of high frequency dispersive estimate:
From the subordination formula and spectral theory, there exist functions ρ, a and η as in the subordination formula so that

$$\begin{aligned}\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)} &= \rho_t(L,\lambda) + \sqrt{t\lambda^{2m_1}}\eta(\lambda^{-2}L)\int e^{it\lambda^{2m_1-2}sL}a_t(s,\lambda)ds \\ &= \rho_t(L,\lambda) + A_{t,\lambda}(L).\end{aligned}$$

We first estimate the term related to $A_{t,\lambda}(L)$. By using (A1), (A2) and estimates on a_t we have

$$\begin{aligned}\|A_{t,\lambda}(L)\|_{L^1\to L^\infty} &\lesssim \sqrt{\lambda^{2m_1}t}[\lambda^{2m_1-2}t]^{-n/2}\int_{2c_0^{-1}}^{2c_0} s^{-n/2}|a_t(s,\lambda)|ds \\ &\lesssim t^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}.\end{aligned}$$

We now take care of the term $\rho_t(L,\lambda)$. Let $\varphi \in C^\infty(\mathbb{R})$ with supp $\varphi \subset [1/6, 6]$ and $\varphi \equiv 1$ in $[1/5, 5]$. Since $\rho_t(\cdot,\lambda)$ is supported in $[\lambda^2/5, 5\lambda^2]$, we have

$$\rho_t(L,\lambda) = \varphi(\lambda^{-1}\sqrt{L})\rho_t(L,\lambda)\varphi(\lambda^{-1}\sqrt{L}).$$

Therefore,

$$\|\rho_t(L,\lambda)\|_{L^1\to L^\infty} \le \|\varphi(\lambda^{-1}\sqrt{L})\|_{L^1\to L^2}\|\rho_t(L,\lambda)\|_{L^2\to L^2}\|\varphi(\lambda^{-1}\sqrt{L})\|_{L^2\to L^\infty}.$$

We now estimate $\|\varphi(\lambda^{-1}\sqrt{L})\|_{L^1\to L^2}$ and $\|\varphi(\lambda^{-1}\sqrt{L})\|_{L^2\to L^\infty}$ by using the Gaussian upper bound and obtain

$$\|\varphi(\lambda^{-1}\sqrt{L})\|_{L^1\to L^2} \lesssim \lambda^{n/2}, \quad \text{and} \quad \|\varphi(\lambda^{-1}\sqrt{L})\|_{L^2\to L^\infty} \lesssim \lambda^{n/2}. \tag{9}$$

By using the properties of $\rho_t(x,\lambda)$ in the subordination formula, we deduce that

$$\|\rho_t(L,\lambda)\|_{L^2\to L^2} \le \|\rho_t(\cdot,\lambda)\|_{L^\infty} \lesssim (\lambda^{2m_1}t)^{-\frac{n-1}{2}}.$$

Therefore,

$$\|\rho_t(L,\lambda)\|_{L^1\to L^\infty} \lesssim \lambda^n(\lambda^{2m_1}t)^{-\frac{n-1}{2}} = t^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}.$$

Summing up, we have proved that

$$\|\psi(\lambda^{-1}\sqrt{L})e^{it\phi(L)}\|_{L^1\to L^\infty} \lesssim t^{-\frac{n-1}{2}}\lambda^{(1-m_1)n+m_1}, \quad \lambda \ge 1. \tag{10}$$

This completes our proof.

A case study: dispersive estimates for fractional Schrödinger semigroups

Theorem ([BDDM])

Let L satisfy (A1) and (A2), and let $\nu \in (0,1)$. Assume that $\psi \in C^\infty(\mathbb{R})$ is supported in $[1/2,2]$. Then we have

$$|\psi(\lambda^{-1}\sqrt{L})e^{itL^\nu} f| \lesssim |t|^{-\frac{n-1}{2}} \lambda^{(1-\nu)n+\nu} \|f\|_{L^1}, \quad \lambda > 0, \ |t| < T_0. \qquad (11)$$

• In the classical case $L = -\Delta$, the decay rate in (11) $\sim t^{-\frac{n-1}{2}}$ as $\nu = 1/2$ and $\sim t^{-\frac{n}{2}}$ as $\nu \neq 1/2$.

• The estimate (11) is sharp in the sense that for each $\nu \in (0,1), \nu \neq 1/2$ we can construct an operator L such that e^{itL} has a decay rate $\sim t^{-\frac{n}{2}}$ while e^{itL^ν} decays like $\sim t^{-\frac{n-1}{2}}$ at best in general.

We have the following theorem.

Theorem ([BDDM])

Let L satisfy (A1) and (A2), and let $\nu \in (0,1)$.

(i) *For $s > (1-\nu)n + \nu$,*

$$\|e^{itL^{\nu}} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I+L)^{s/2} f\|_{L^1}, \quad |t| < T_0.$$

(ii) *For $p \in (0,1)$ and $s = n(1/p - \nu) + \nu$,*

$$\|e^{itL^{\nu}} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p_L}, \quad |t| < T_0.$$

Remarks:

(a) The estimate (ii) is new. To the best of our knowledge, this is the first $H^p_L - L^\infty$ dispersive estimate in the literature.

(b) In the particular case when $L = -\Delta$, the estimates (i) and (ii) can be improved as follows: for all $\nu \in (0,1)\backslash\{1/2\}$,

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \|(I+L)^{s/2} f\|_{L^1}, \quad s > n(1-\nu), \tag{12}$$

and

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \|L^{s/2} f\|_{H^p}, p \in (0,1], s = n(1/p-\nu), \tag{13}$$

where H^p is a classical Hardy space. To the best of our knowledge, the estimates (12) and (13) are new.

We have the following result for the case $\nu = 1/2$.

Theorem ([BDDM])

Let L satisfy (A1) and (A2).

(i) *For $s > \frac{n-1}{2}$, we have*

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I+L)^{s/2} f\|_{L^1}, \quad |t| < T_0.$$

(ii) *For $p \in (0,1]$ and $s = n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$, we have*

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p_L}, \quad |t| < T_0.$$

Note that the decay $|t|^{-\frac{n-1}{2}}$ is sharp. Compare with the following estimate from [Beals, 1994] (see also [B. Marshall, W. Strauss, and S. Wainger, 1980]) for $L = -\Delta + V$ with small potentials $V \in S(\mathbb{R}^n), n \geq 3$:

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|(I - \Delta)^{s/2} f\|_{H^1}, \ s = \frac{n-1}{2}. \tag{14}$$

We see that estimate (ii) is new for $0 < p \leq 1$ even when $L = -\Delta$.

The following estimate concerns the Besov norm associated to operator L.

Theorem ([BDDM])

Let L satisfy (A1) and (A2), and let $\nu \in (0,1)$. Then we have

$$\left\| e^{itL^{\nu}} f \right\|_{L^{\infty}} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{(1-\nu)n+\nu,L}}. \tag{15}$$

In the particular case $\nu = \frac{1}{2}$ we get

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^{\infty}} \lesssim |t|^{-\frac{n-1}{2}} \|f\|_{\dot{B}_{1,1}^{\frac{n-1}{2},L}}. \tag{16}$$

Applications

1. Hermite operators. Let $L = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^n$ with $n \geq 1$. It is well-known that for any $\delta > 0$ there exists $C > 0$ so that

$$\|e^{itL}\|_{L^1 \to L^\infty} \leq \frac{C}{t^{n/2}}, \quad |t| < \pi/2 - \delta.$$

Theorem ([BDDM])

Let $L = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^n$ with $n \geq 1$. Then we have

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p_L}, \quad |t| < \pi/2 - \delta$$

for $p \in (0,1)$ and $s = n(1/p - \nu) + \nu$; and

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p_L}, |t| < \pi/2 - \delta$$

for $p \in (0,1)$ and $s = n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

We can show that $H^p(\mathbb{R}^n) \subsetneq H^p_L(\mathbb{R}^n)$. See [J. Dziubański, 1998]. Hence the estimate above is sharper than the following estimates:

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p}, \quad |t| < \pi/2 - \delta$$

for $p \in (0, 1)$ and $s = n(1/p - \nu) + \nu$; and

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{n-1}{2}} \|L^{s/2} f\|_{H^p}, \quad |t| < \pi/2 - \delta$$

for $p \in (0, 1)$ and $s = n(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$.

We can prove that $\dot{B}^{0}_{1,1}(\mathbb{R}) \subset \dot{B}^{0,L}_{1,1}(\mathbb{R})$ and $\dot{B}^{s}_{1,1}(\mathbb{R}^n) \equiv \dot{B}^{s,L}_{1,1}(\mathbb{R}^n)$ for $0 < s < 2$ and $n \geq 1$. Hence, from the estimate

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim t^{-\frac{n-1}{2}} \|f\|_{\dot{B}^{\frac{n-1}{2},L}_{1,1}} \tag{17}$$

we obtain:

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim t^{-\frac{n-1}{2}} \|f\|_{\dot{B}^{\frac{n-1}{2}}_{1,1}}, \quad n = 1, 2, 3, 4.$$

2. Laguerre operators: Consider the space $X = (0, \infty)^n$ equipped with the Euclidean distance d and measure μ given by $d\mu(x) = d\mu_1(x) \dots d\mu_n(x)$ where $d\mu_k = x_k^{2\alpha_k+1} dx_k, \alpha_k > -1$ for $k = 1, \dots, n$.
It is easy to see that

$$\mu(B(x,r)) \sim \prod_{k=1}^{n} (r + x_k)^{2\alpha_k+1} r \tag{18}$$

where $B(x,r) = \{y \in X : |x - y| < r\}$ is the ball centered in $x = (x_1, x_2, \dots, x_n)$ with radius r. It follows that the measure μ satisfies the doubling condition (1).
Moreover, if $\alpha_k > -1/2$ for all k, then we have

$$\mu(B(x,r)) \gtrsim r^N, \quad N = 2n + \sum_{k=1}^{n} 2\alpha_k \geq 1$$

for all $x \in X$ and $r > 0$.
We now consider the Laguerre operator L defined by

$$L = -\Delta - \sum_{k=1}^{n} \frac{2\alpha_k + 1}{x_k} \frac{d}{dx_k} + |x|^2. \tag{19}$$

It is well known that the Laguerre operator satisfies (A1) and (A2).

We have the following result concerning Laguerre operator.

Theorem ([BDDM])

Let $\alpha_k > -1/2$ for all $k = 1, \dots, n$ and let L be the Laguerre operator defined by (19). Then we have

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}} \|L^{s/2} f\|_{H^p_L}, \quad |t| < \pi/2 - \delta$$

for $p \in (0,1]$ and $s = N(1/p - \nu) + \nu$; and

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}} \|L^{s/2} f\|_{H^p_L}, |t| < \pi/2 - \delta$$

for $p \in (0,1)$ and $s = N(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$.

Since $H^p_{CW}(X) \subsetneqq H^p_L(X)$ for $\frac{N}{N+1} < p \le 1$ (see [Bui-Duong-Ly, 2016]). Hence the estimates above imply:

$$\|e^{itL^\nu} f\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}} \|L^{s/2} f\|_{H^p_{CW}}, \quad |t| < \pi/2 - \delta$$

for $p \in (\frac{N}{N+1}, 1)$ and $s = N(1/p - \nu) + \nu$; and

$$\left\| \frac{e^{it\sqrt{L}}}{\sqrt{L}} f \right\|_{L^\infty} \lesssim |t|^{-\frac{N-1}{2}} \|L^{s/2} f\|_{H^p_{CW}}, \quad |t| < \pi/2 - \delta$$

for $p \in (\frac{N}{N+1}, 1)$ and $s = N(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$.

THANK YOU!