Vortex cusps

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Motivation Incompressible 2d Euler:

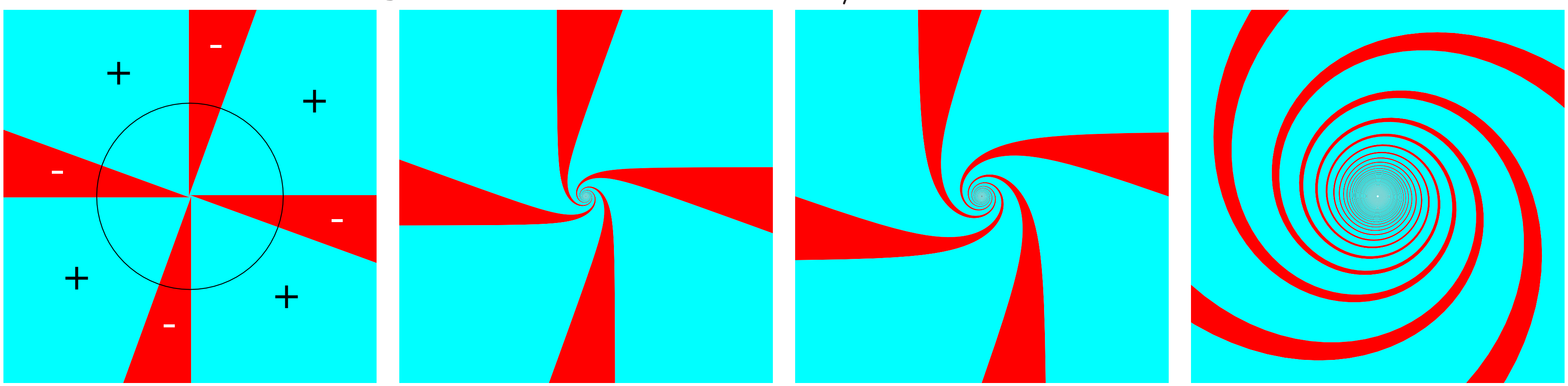
$$\omega_t + \mathbf{v} \cdot \nabla \omega = 0 \quad , \quad \nabla \cdot \mathbf{v} = 0 \quad , \quad \omega = \nabla \times \mathbf{v}$$

Initial data

$$\omega(t = 0, \mathbf{x}) = |\mathbf{x}|^{-1/\mu} \mathring{\omega}(\measuredangle \mathbf{x})$$

Thm [E., Comm. Math. Phys. 2016]: existence of selfsimilar solutions with ω algebraic spiral patterns, if

1. dominant sign: $\int_0^{2\pi} \mathring{\omega} \neq 0$
2. sufficiently high periodicity: $\mathring{\omega}$ $2\pi/N$-periodic with $N \gg 1$

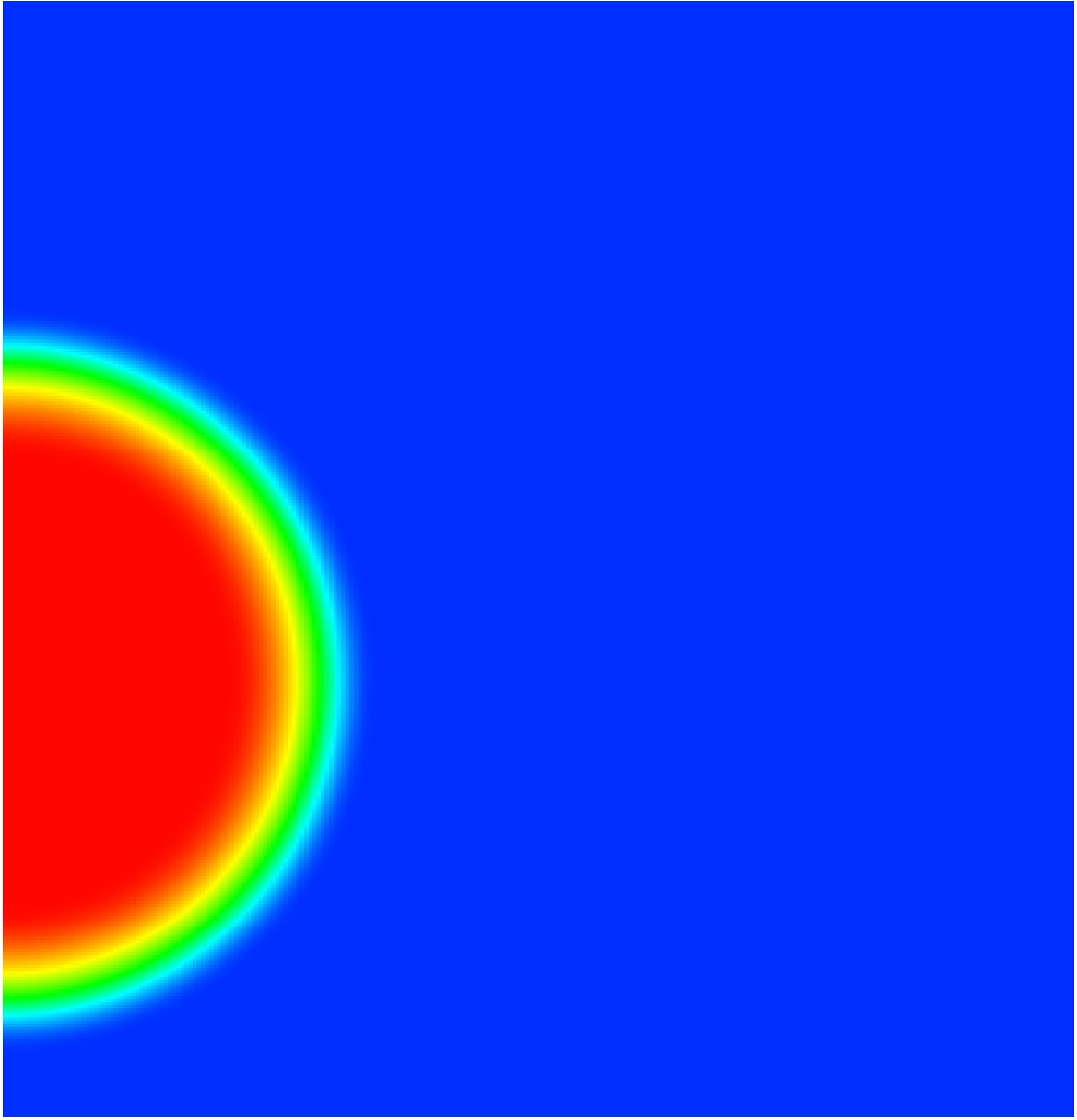


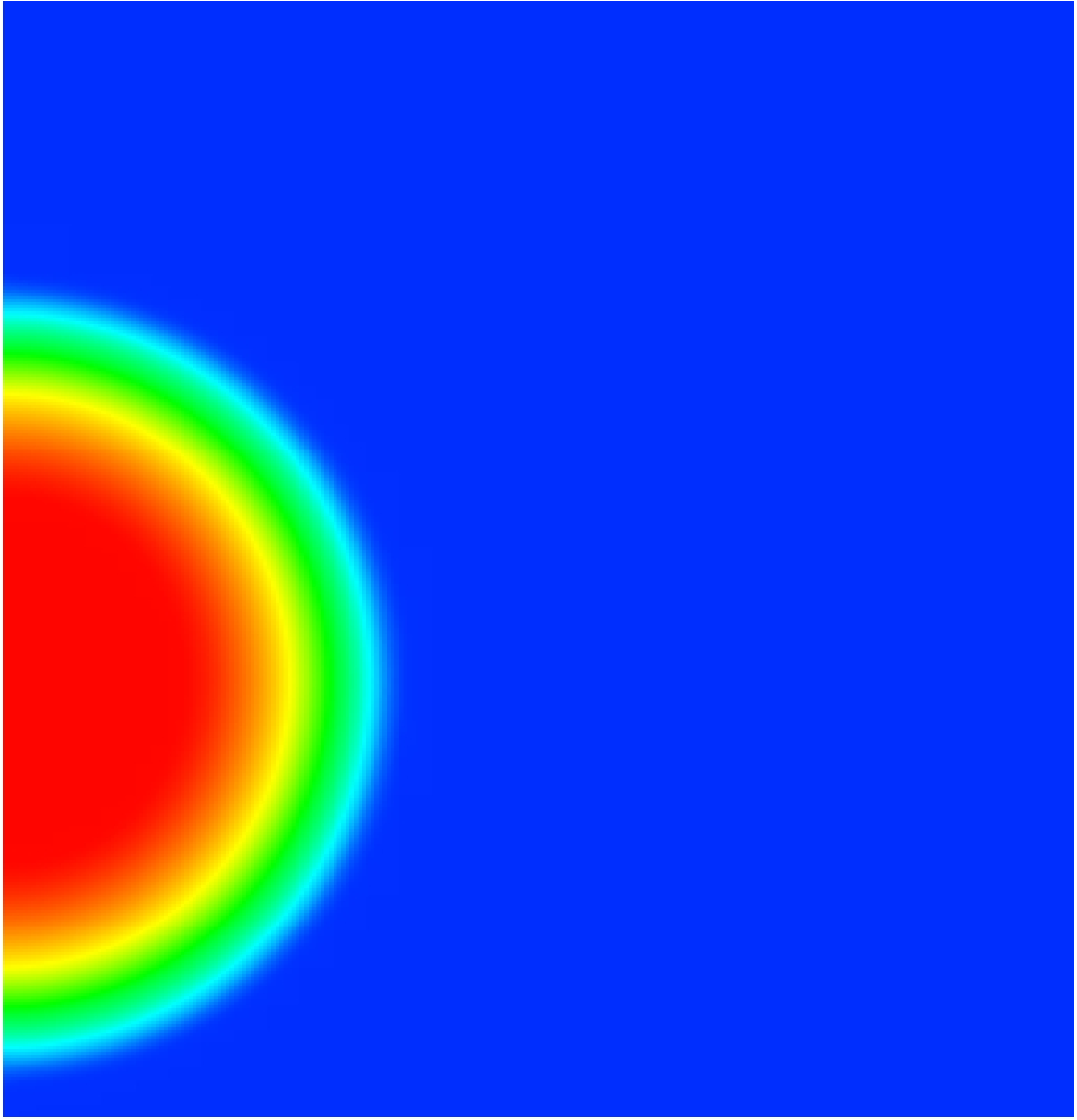
$t = 0$ $\quad$ $t > 0$ $\quad$ $t \gg 0$ $\quad$ $t \ggg 0$

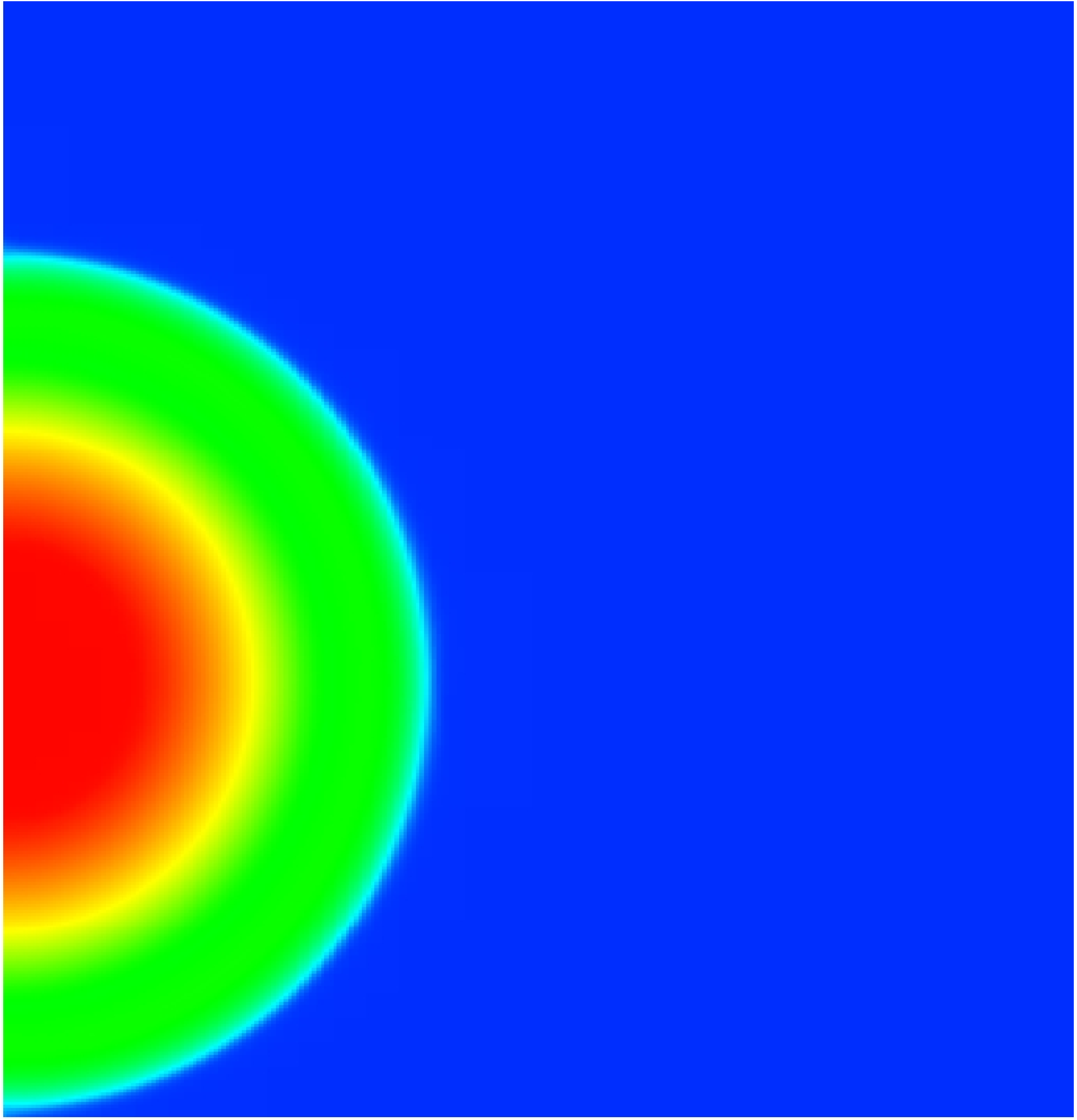
What if neither sign dominates?

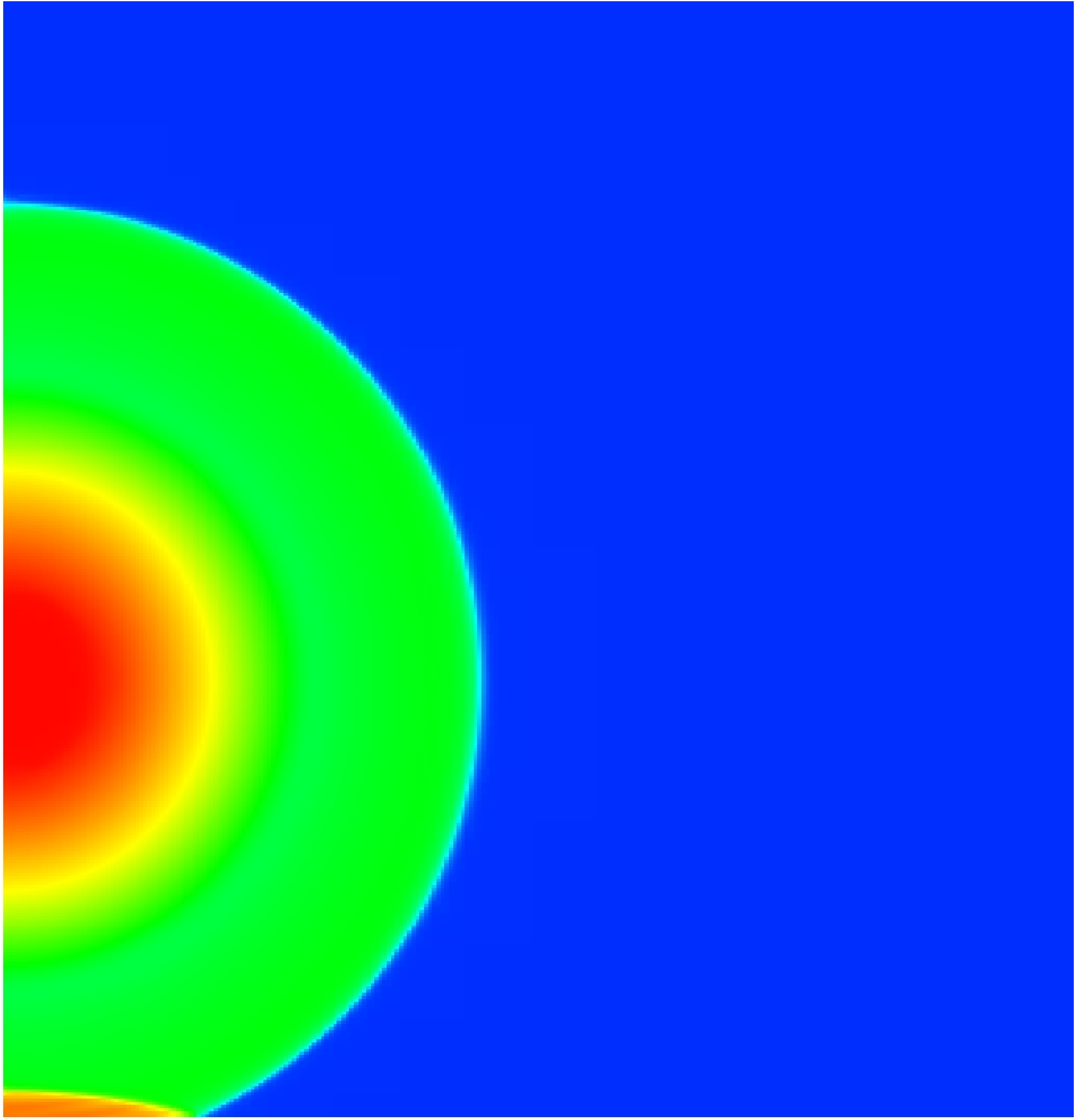
Vorticity generation: Mach reflection

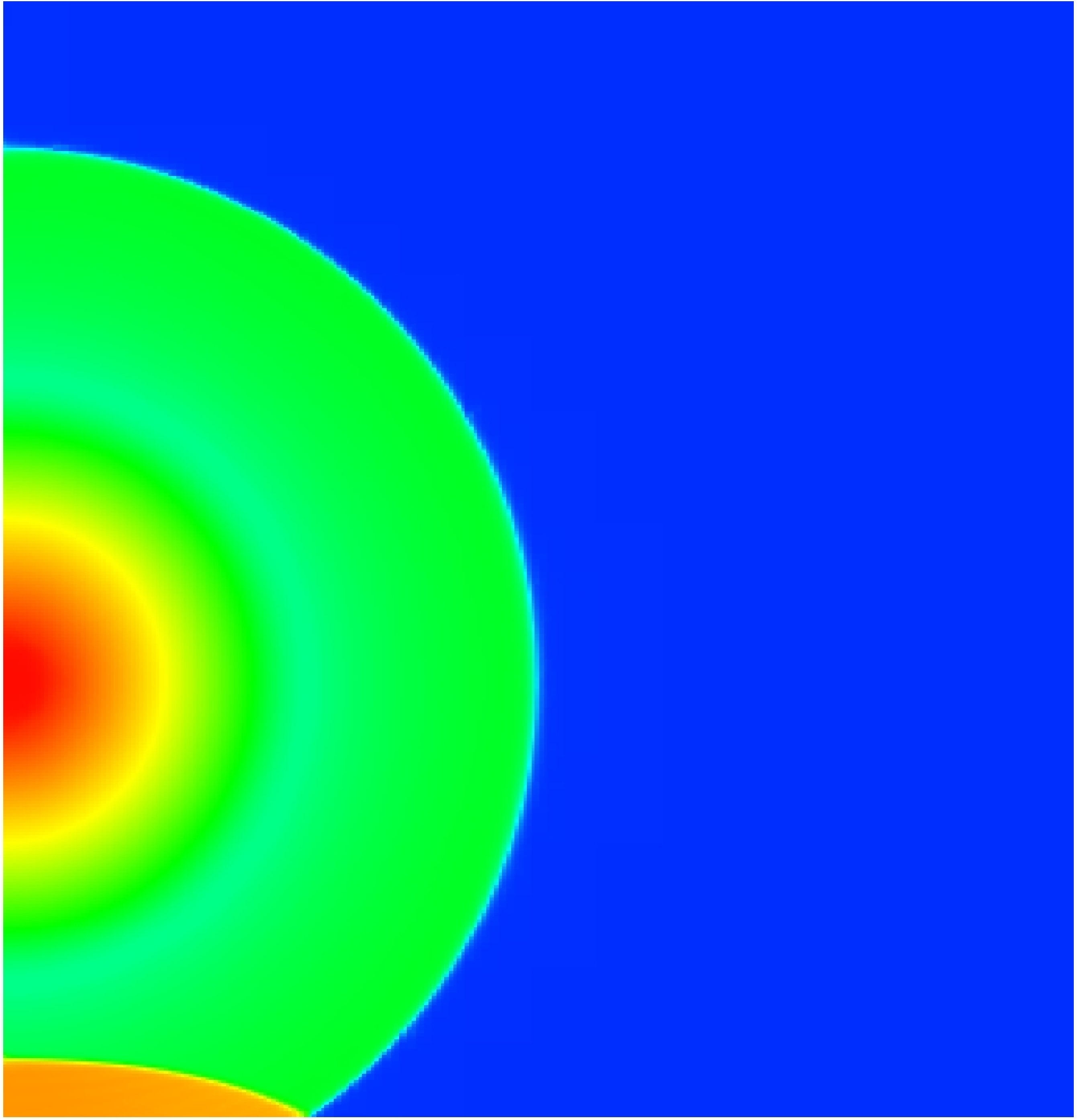
From *smooth* initial data!

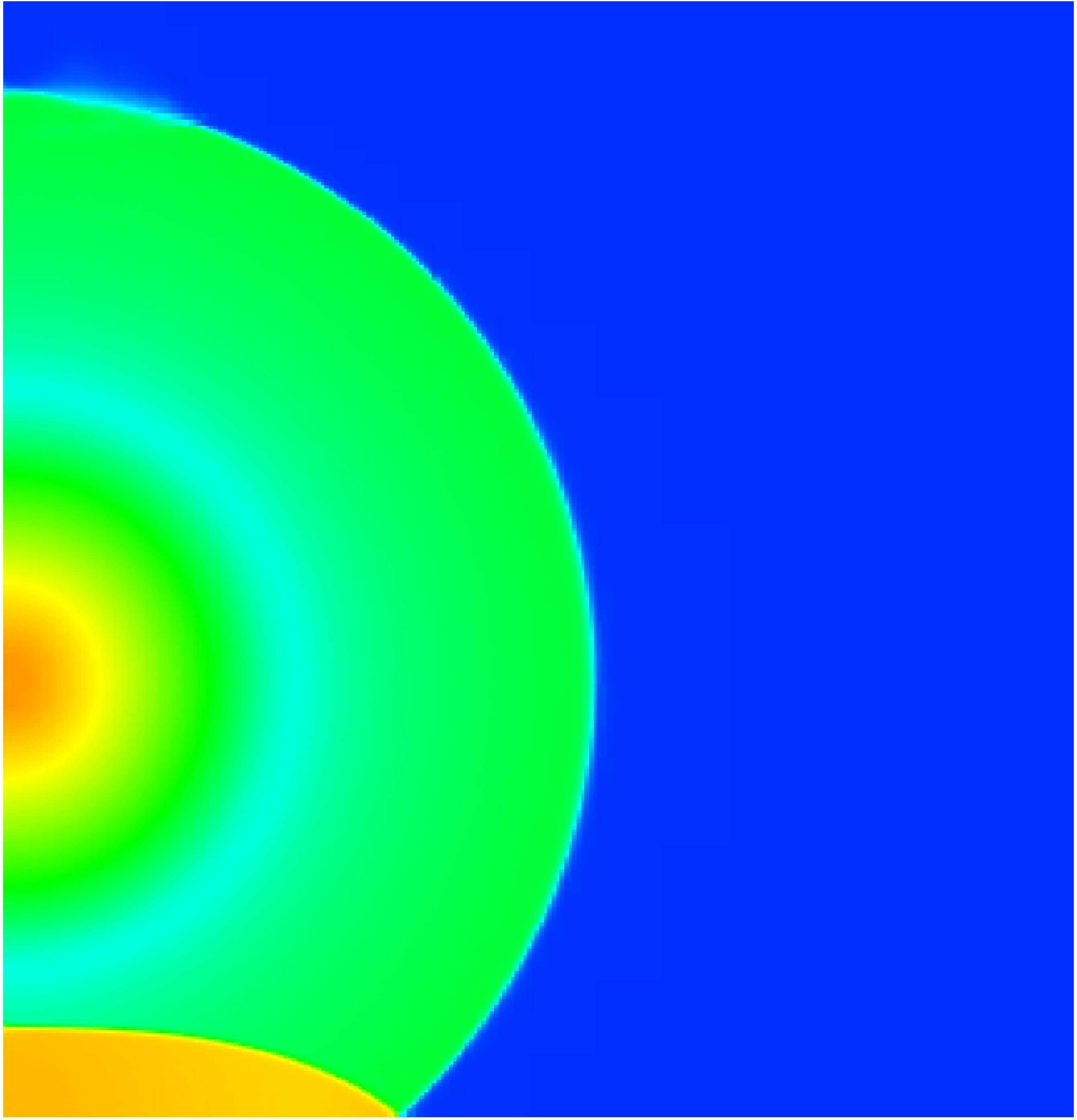


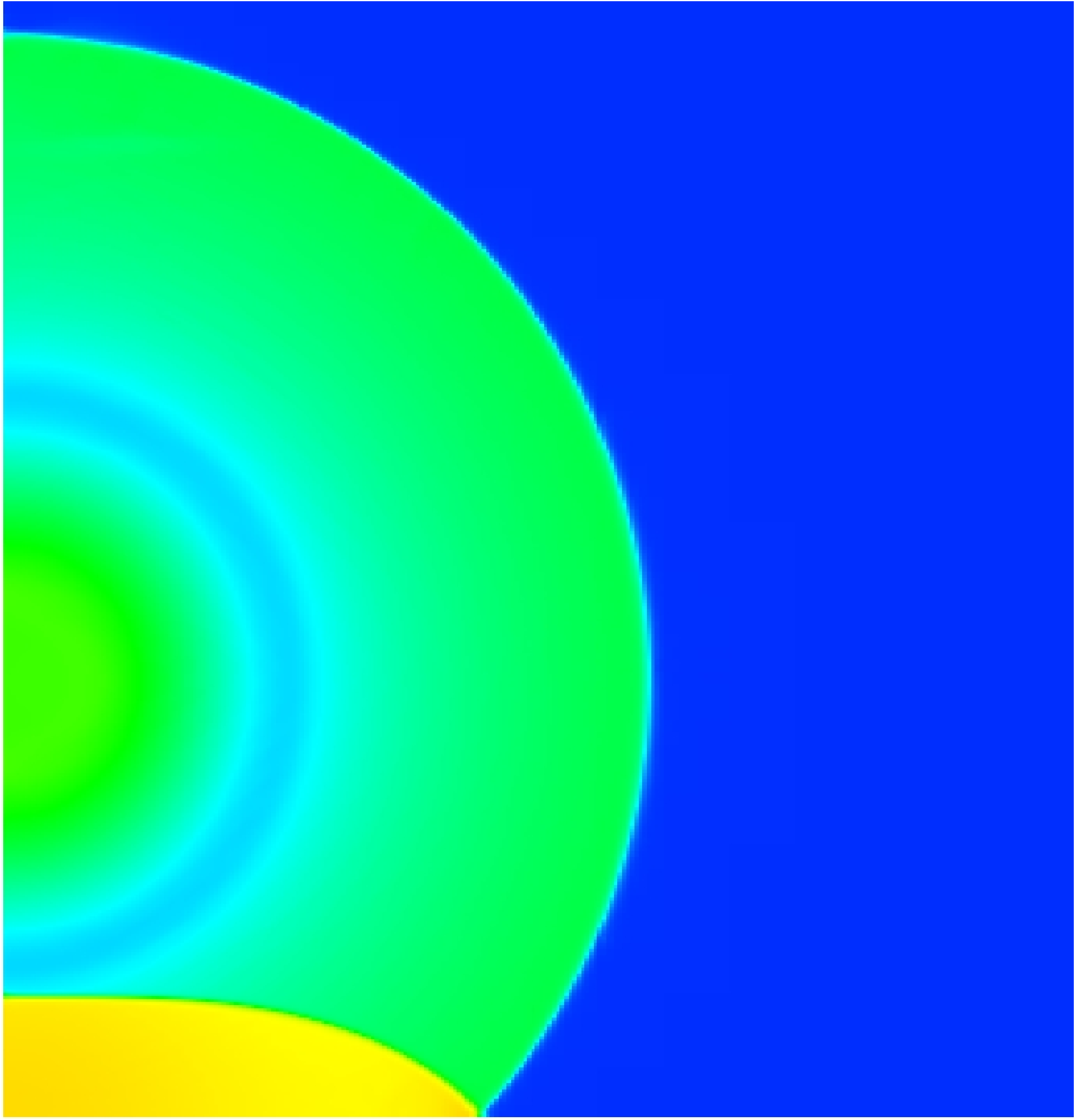


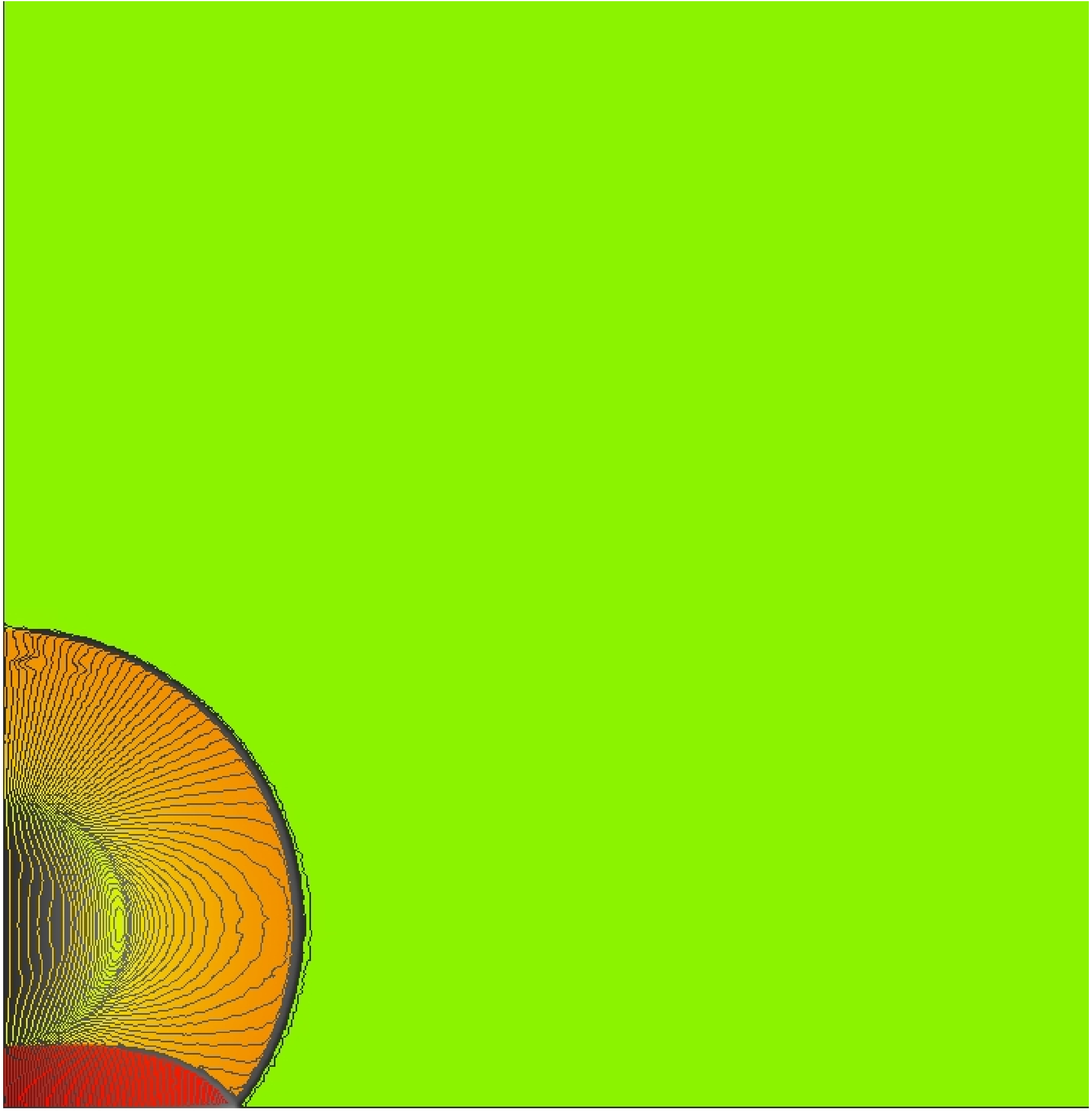


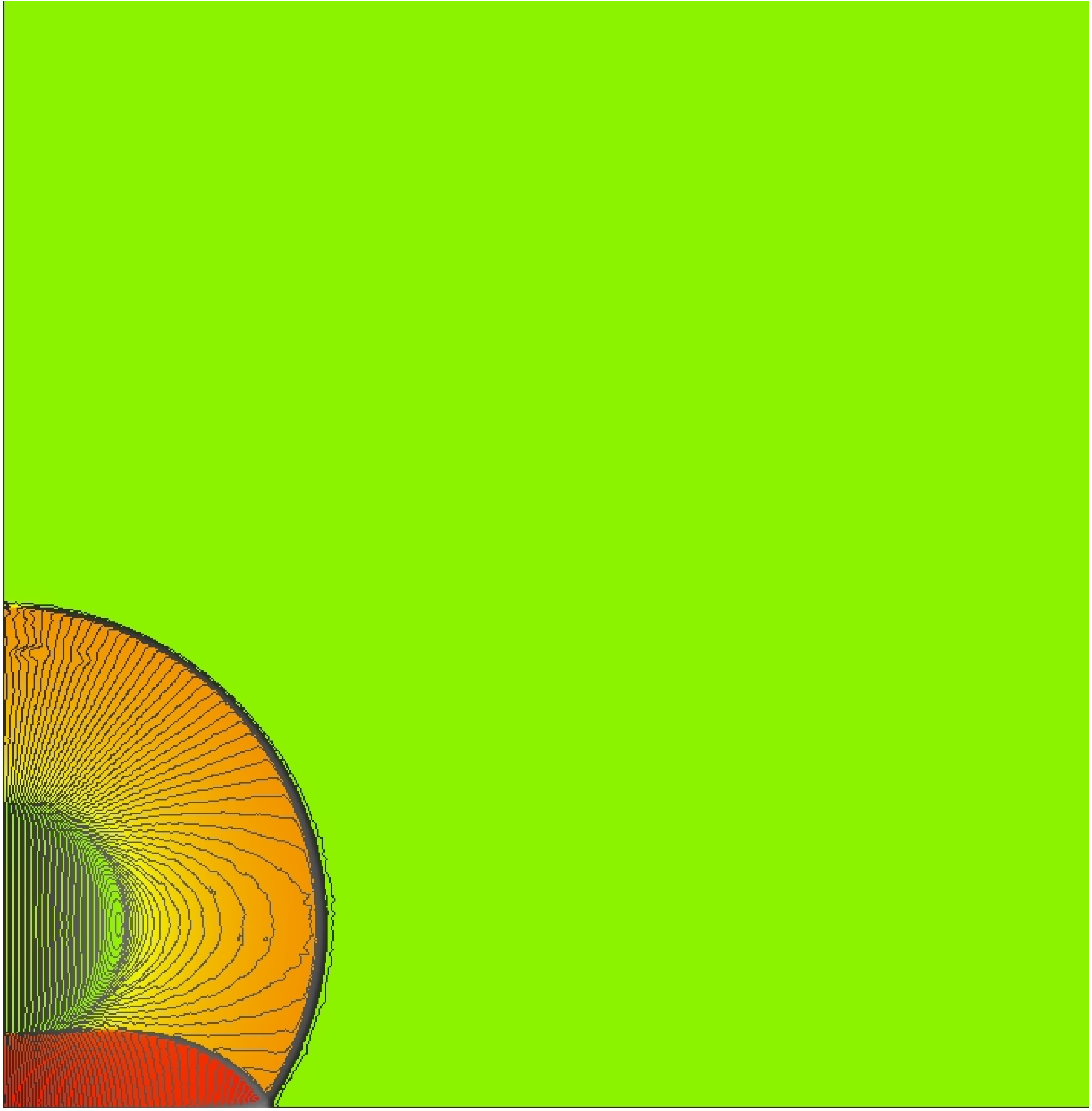


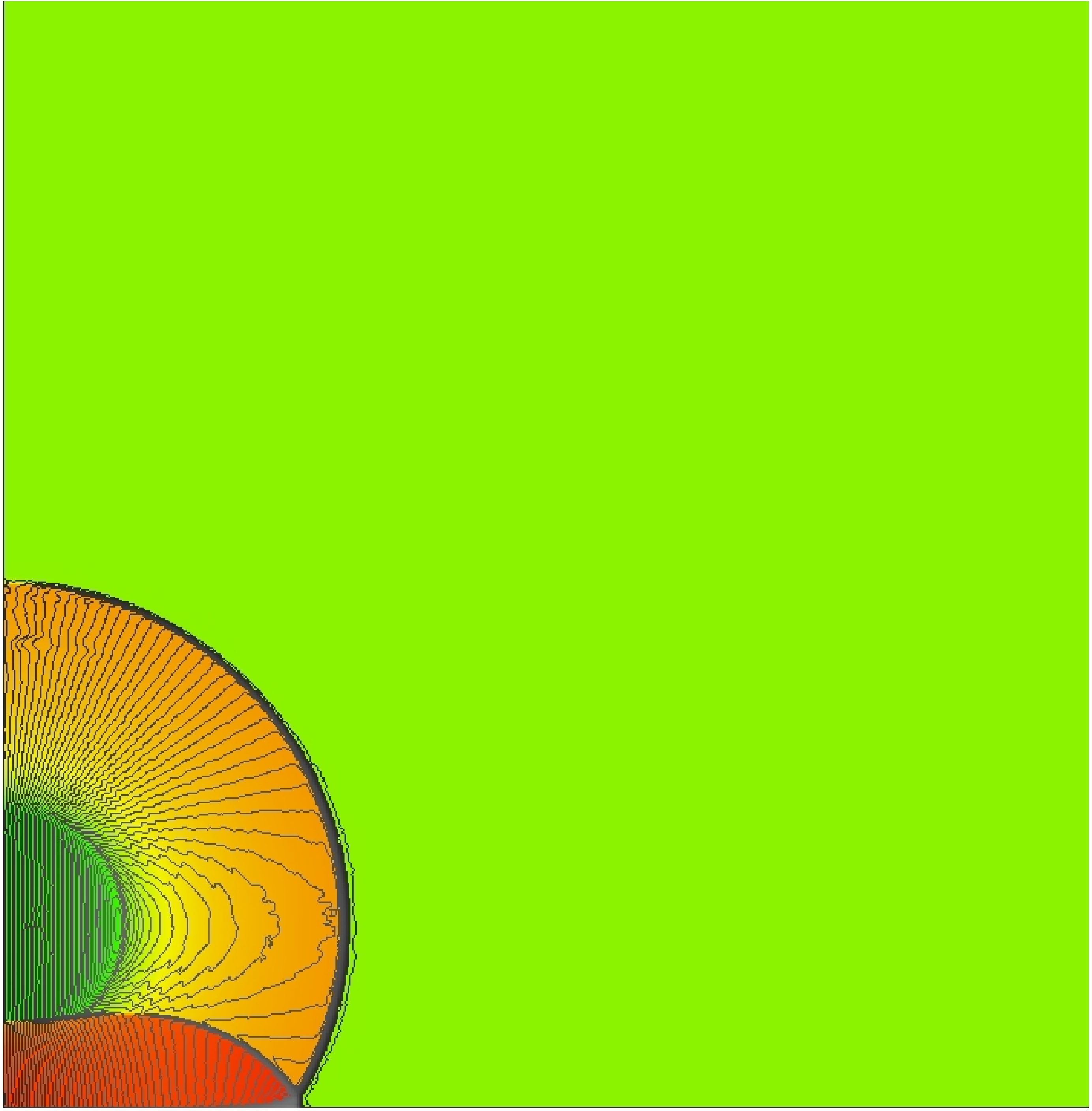


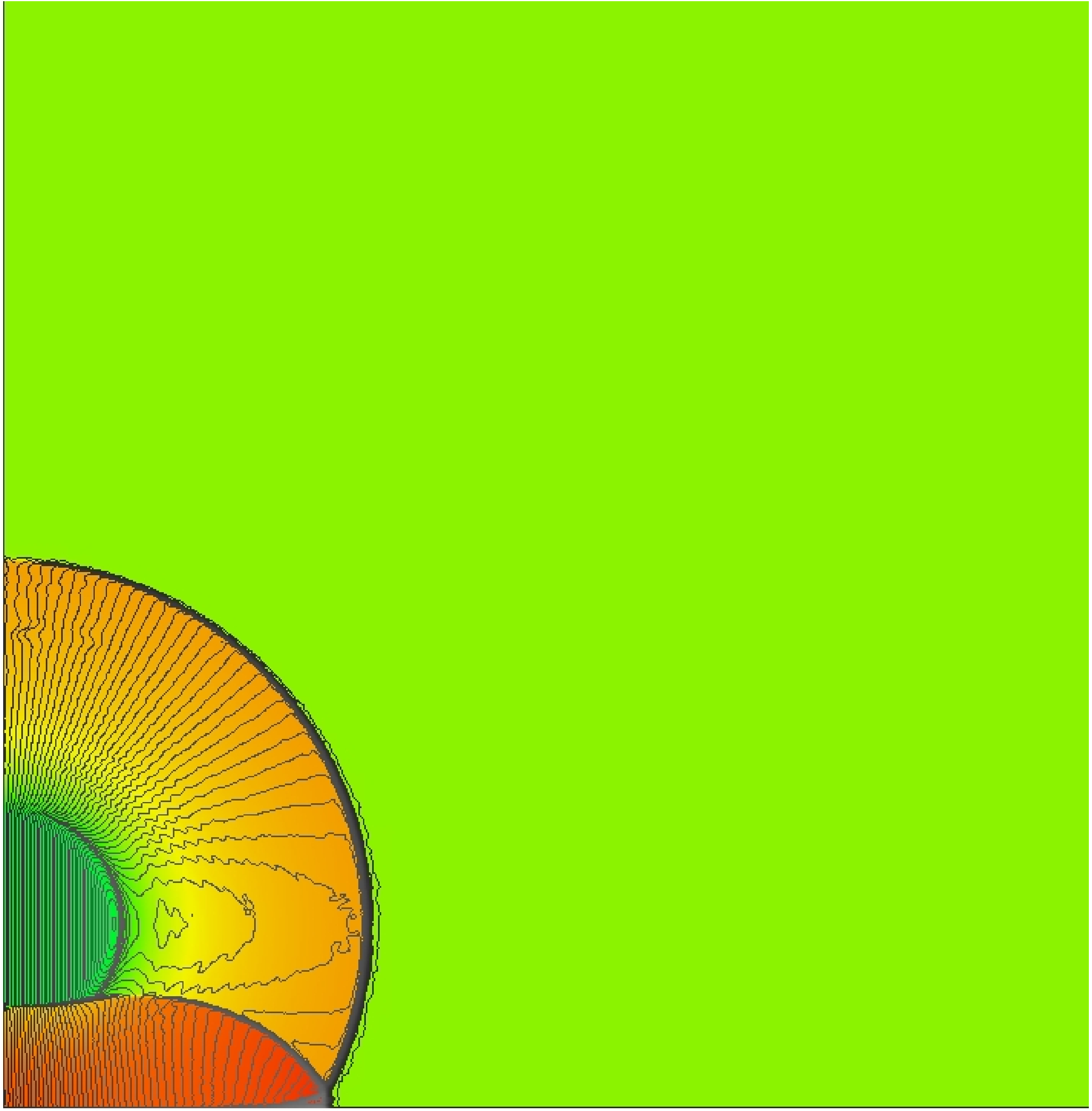


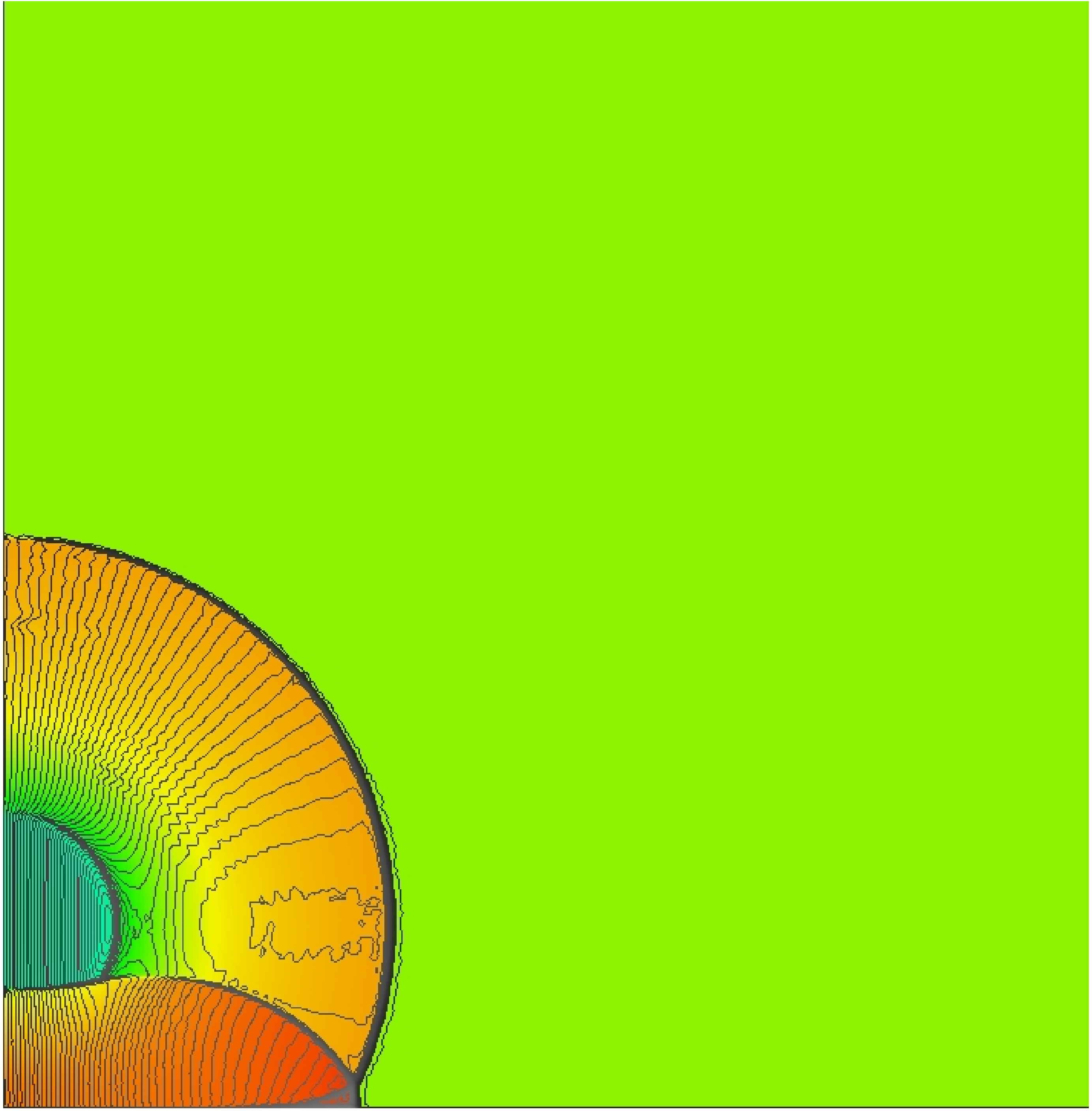


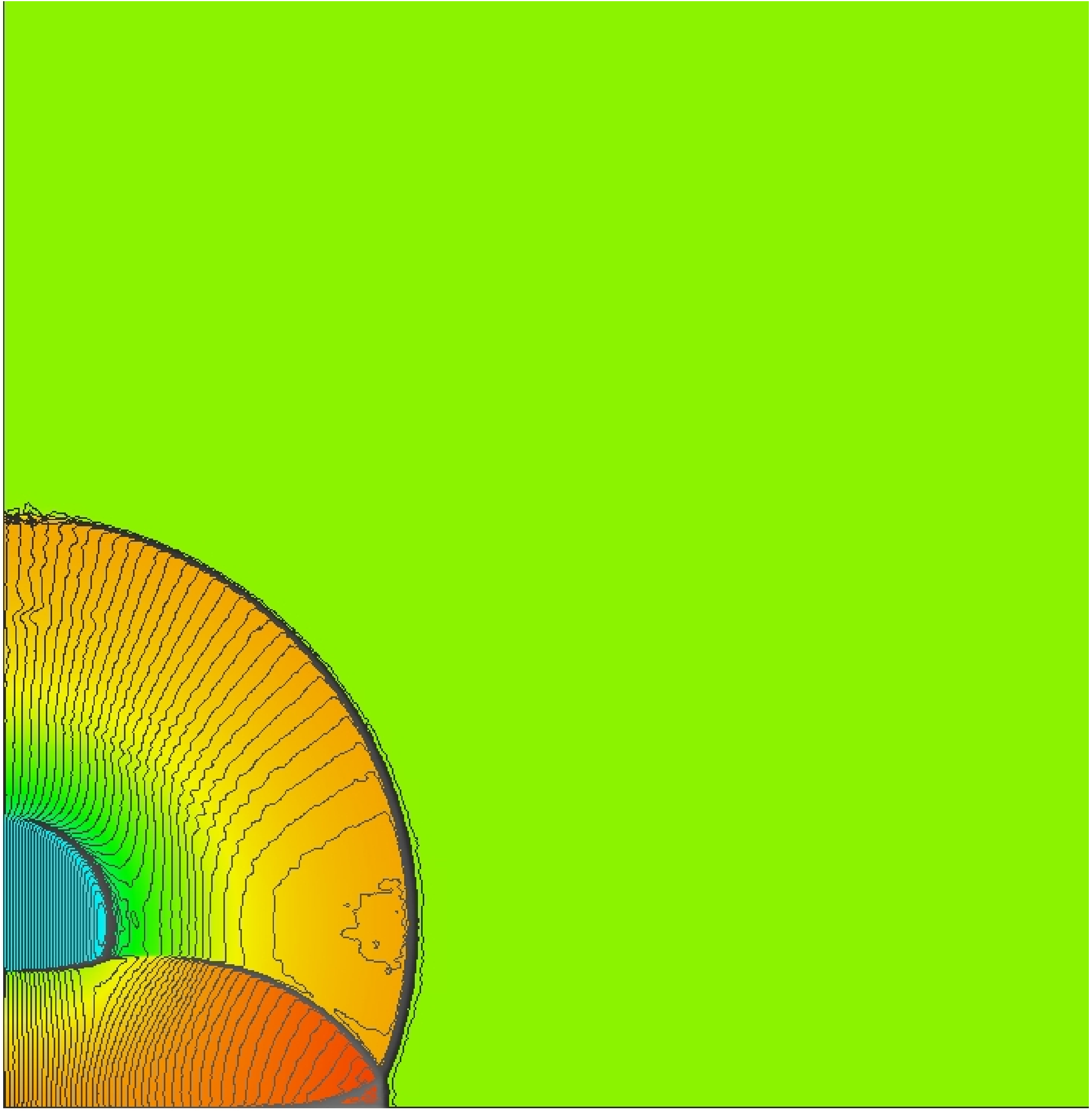


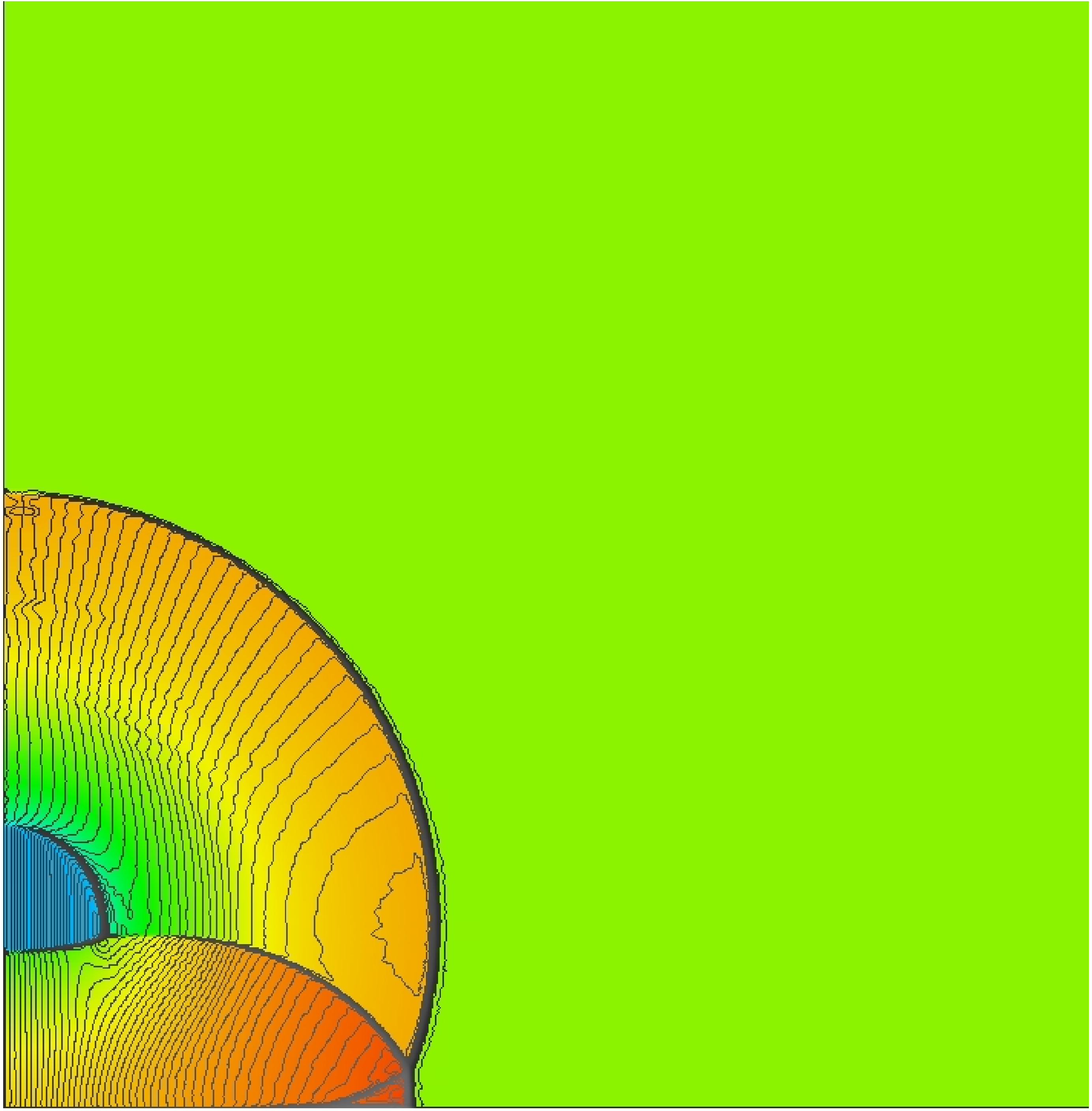


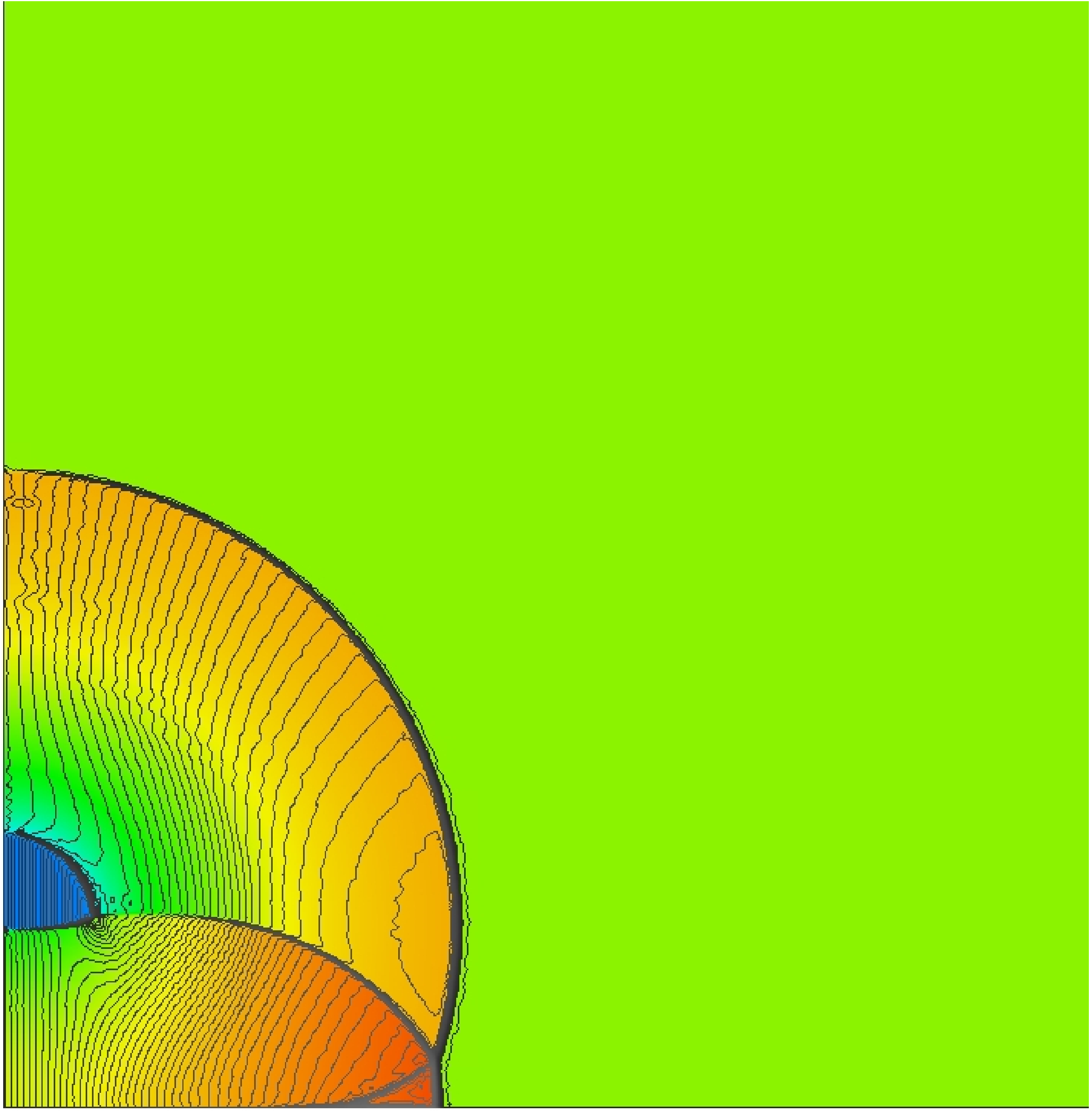


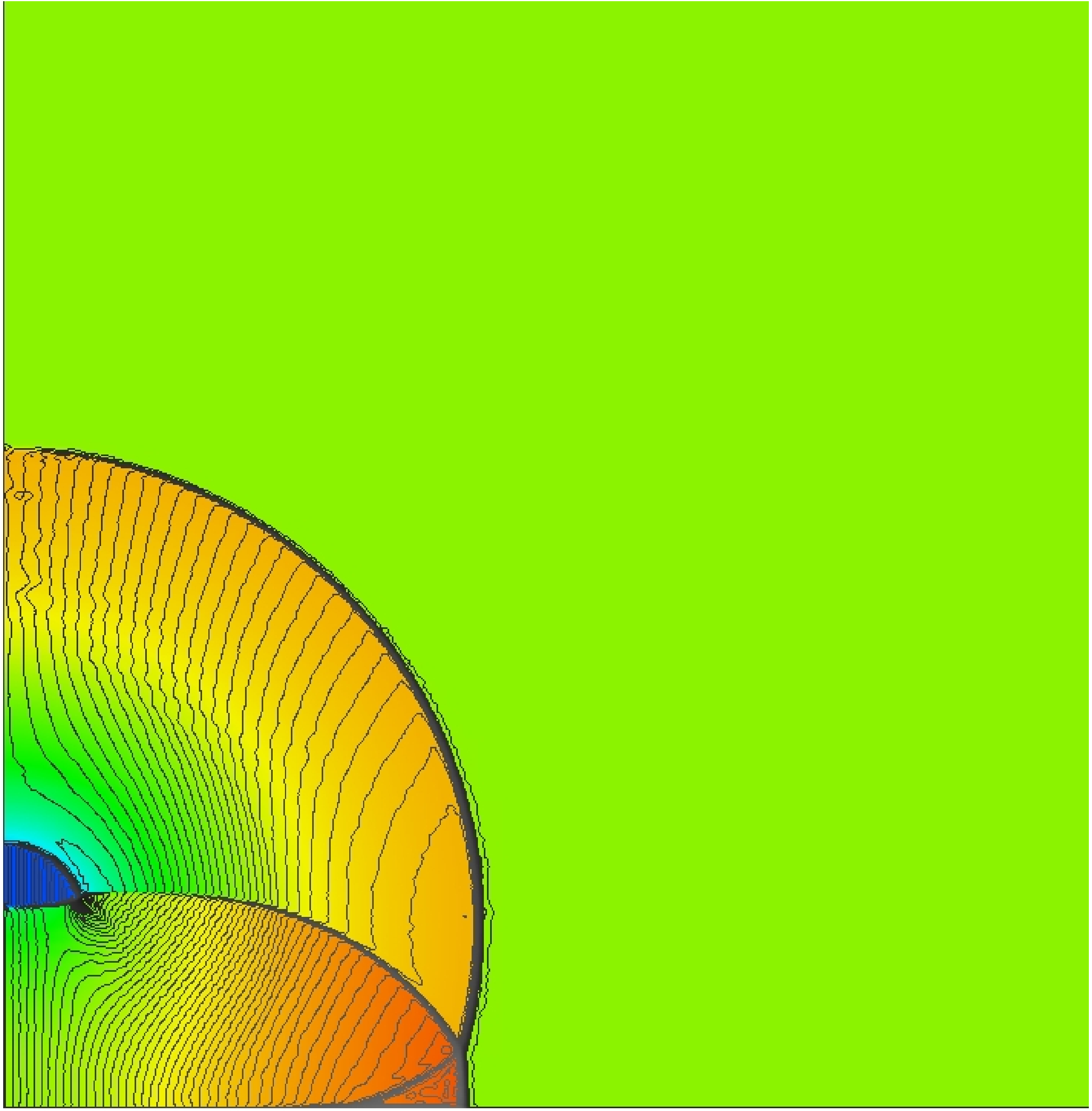


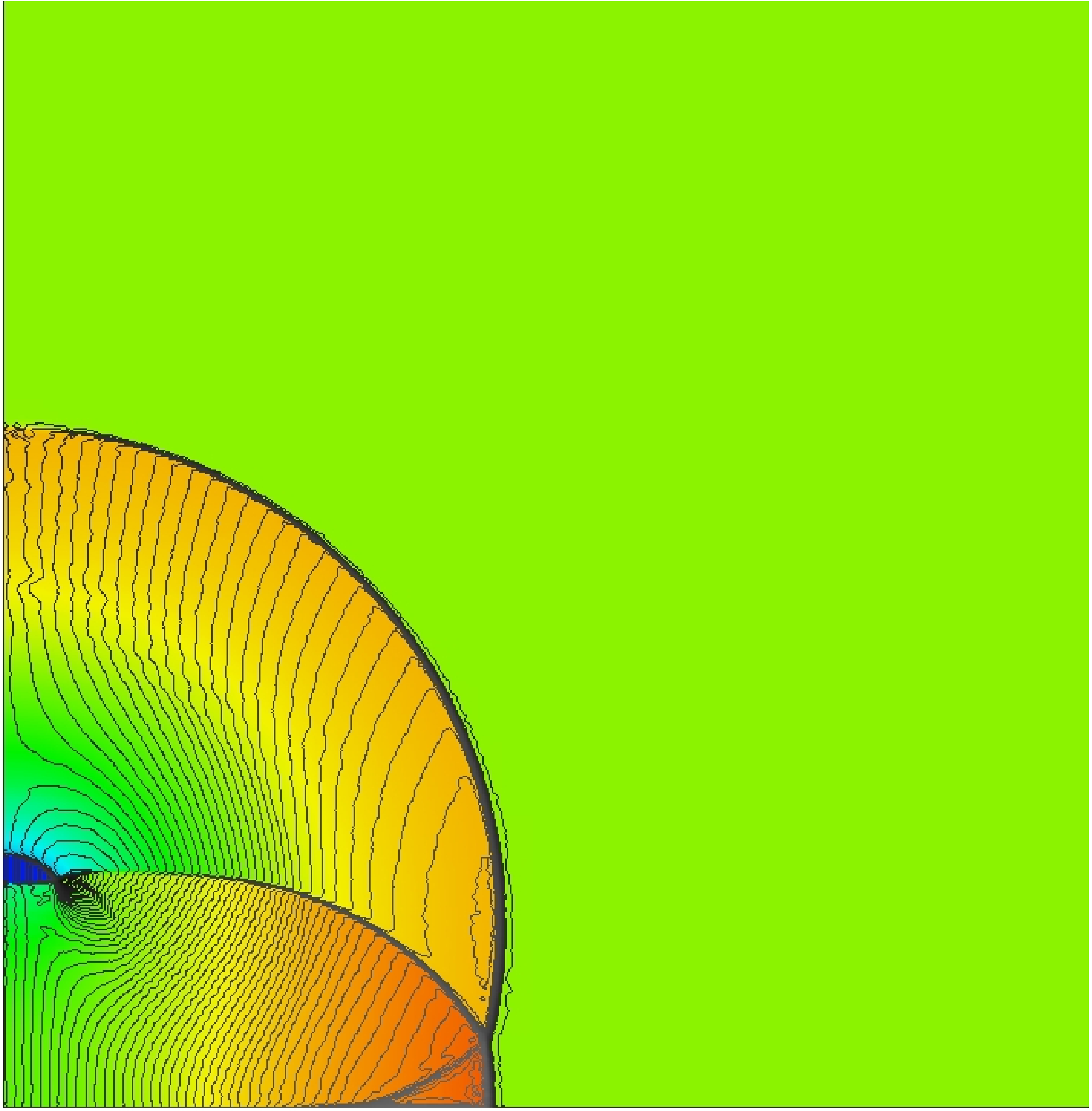


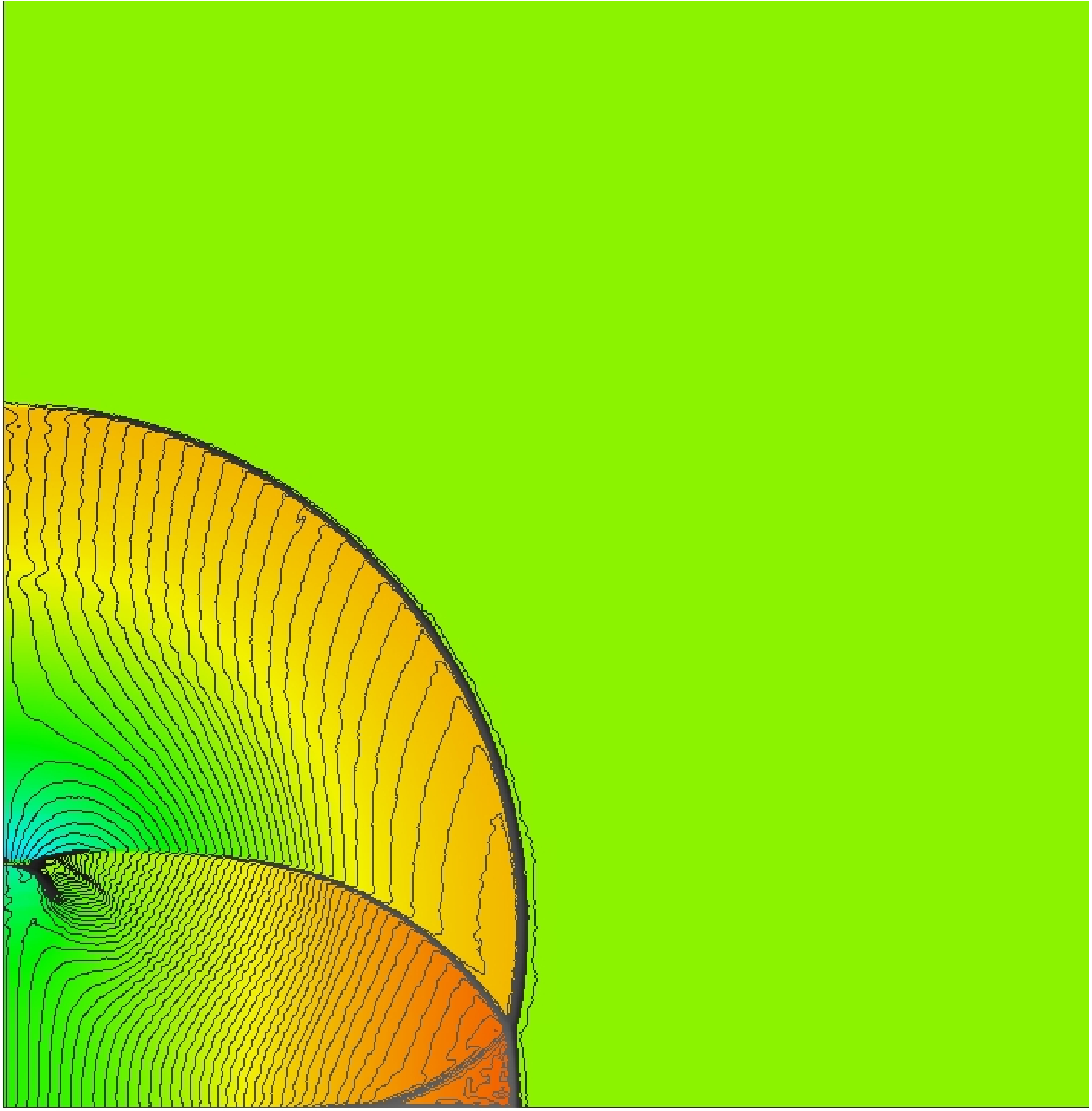


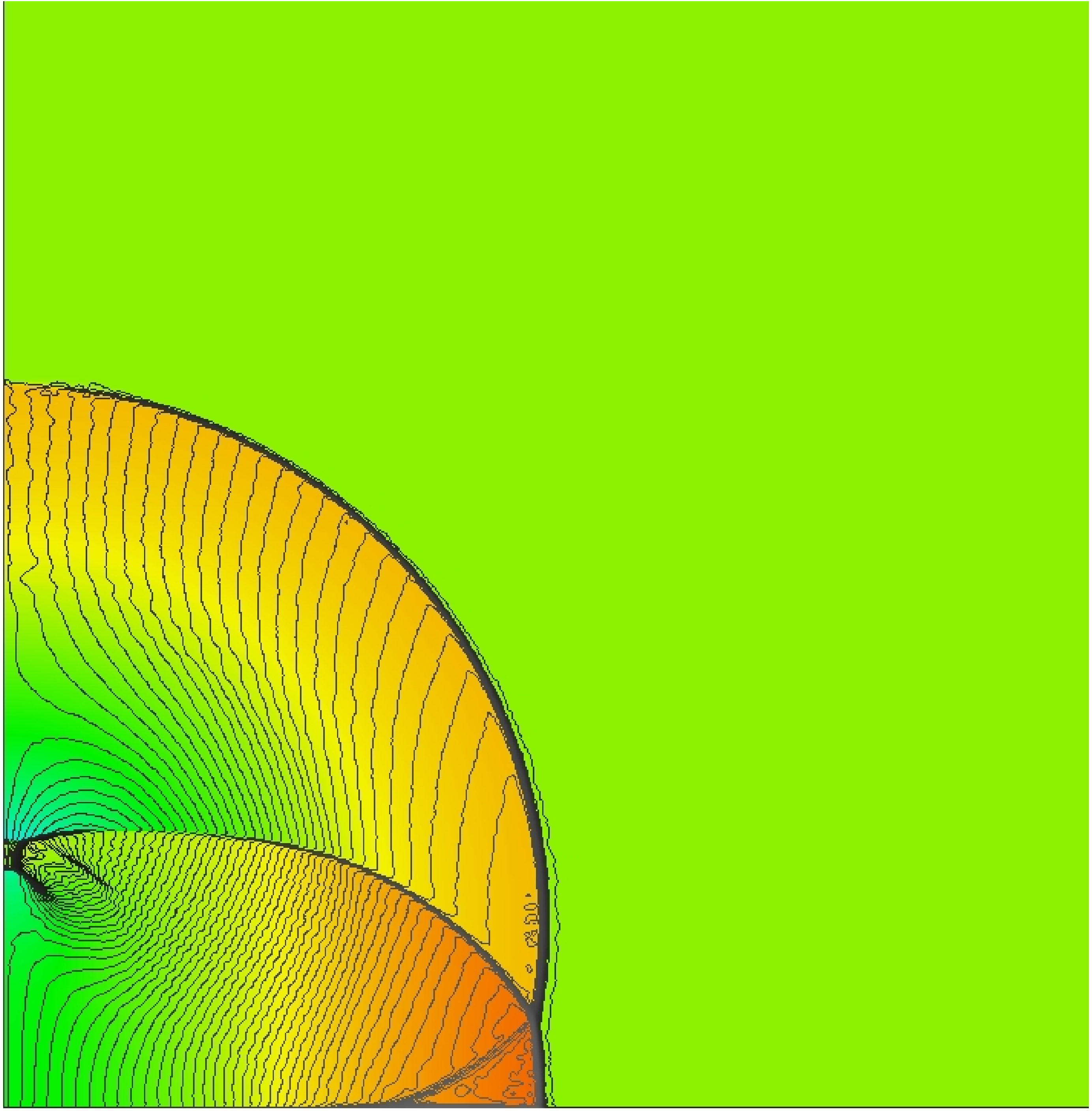


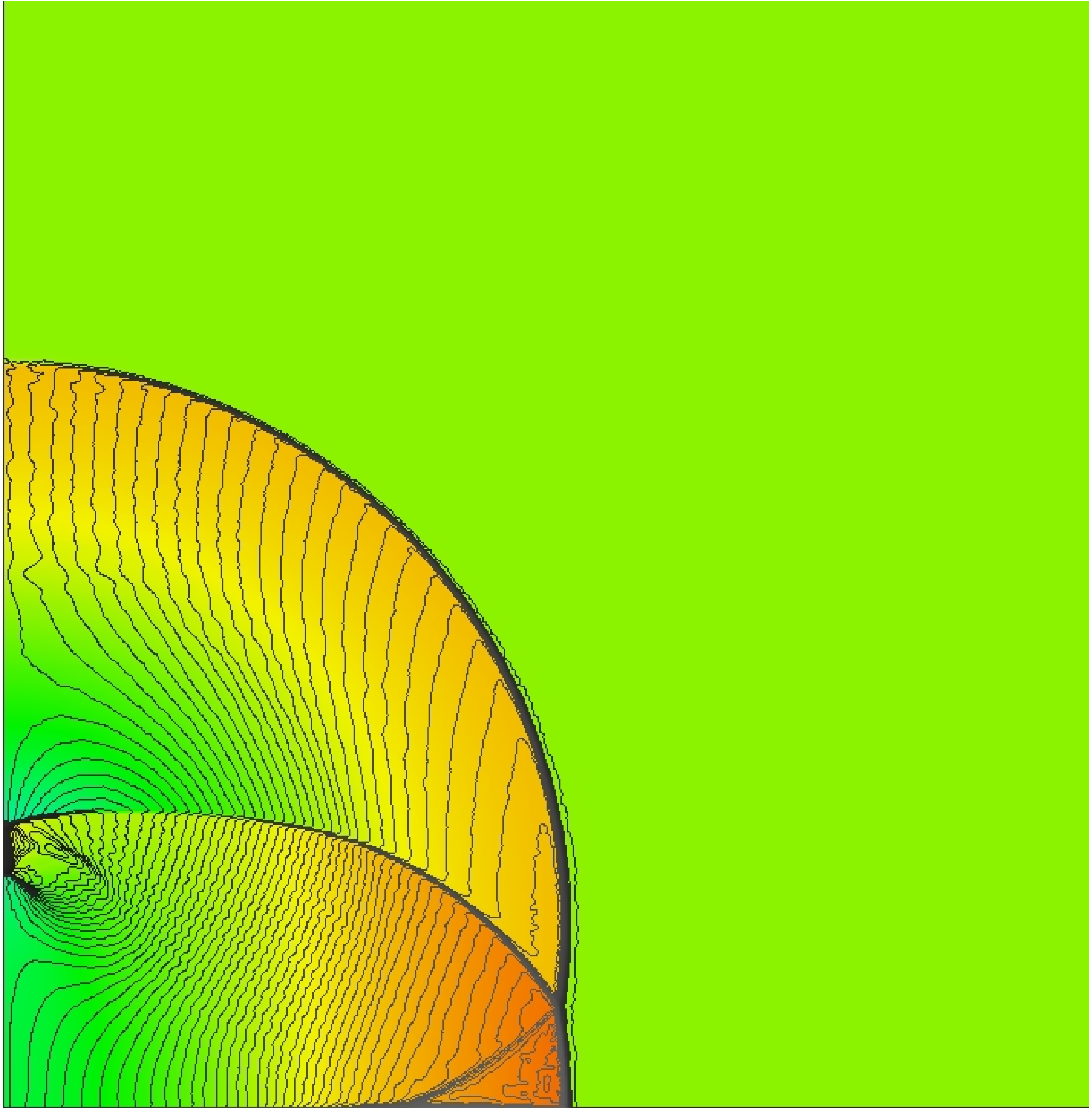


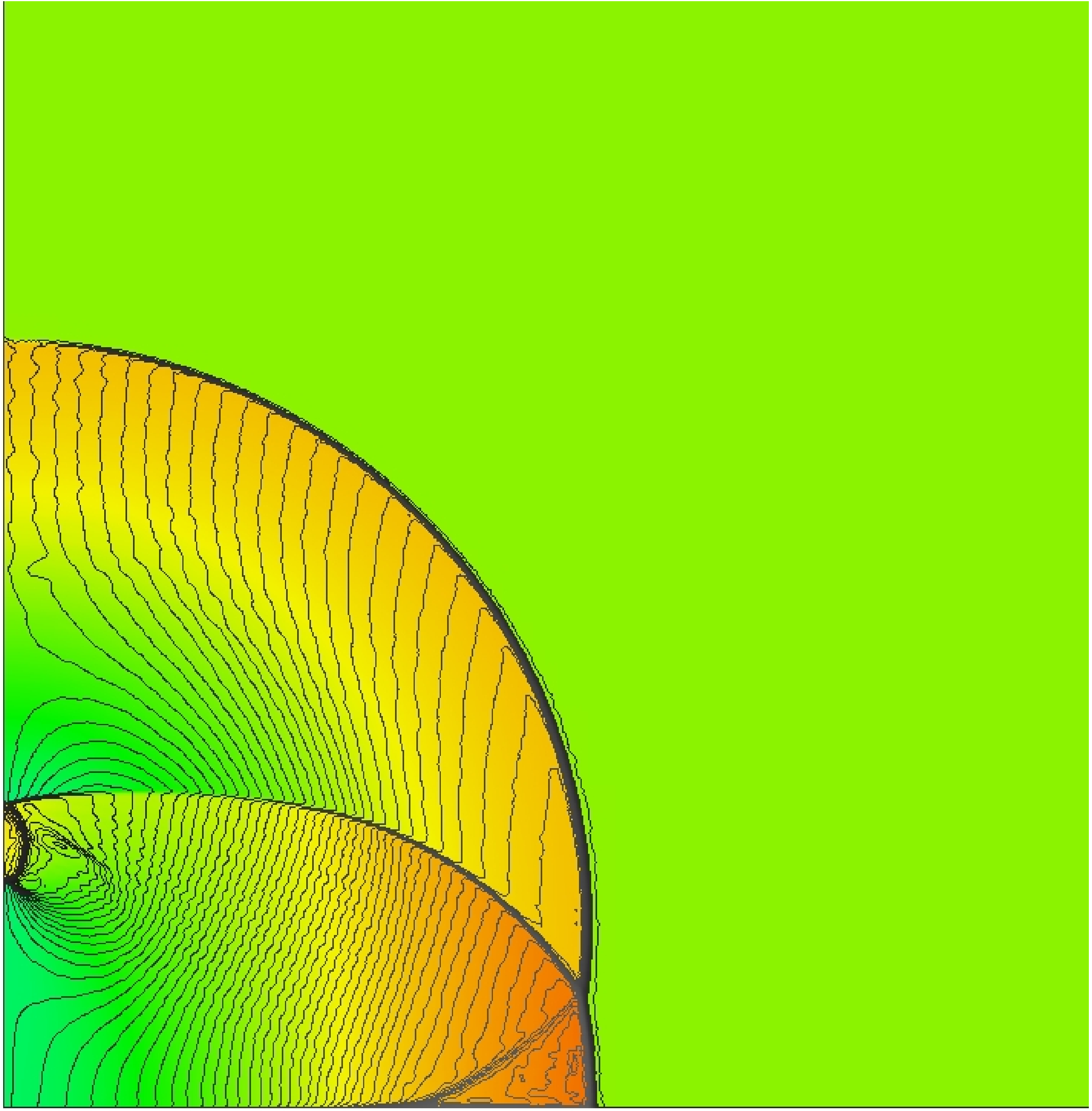


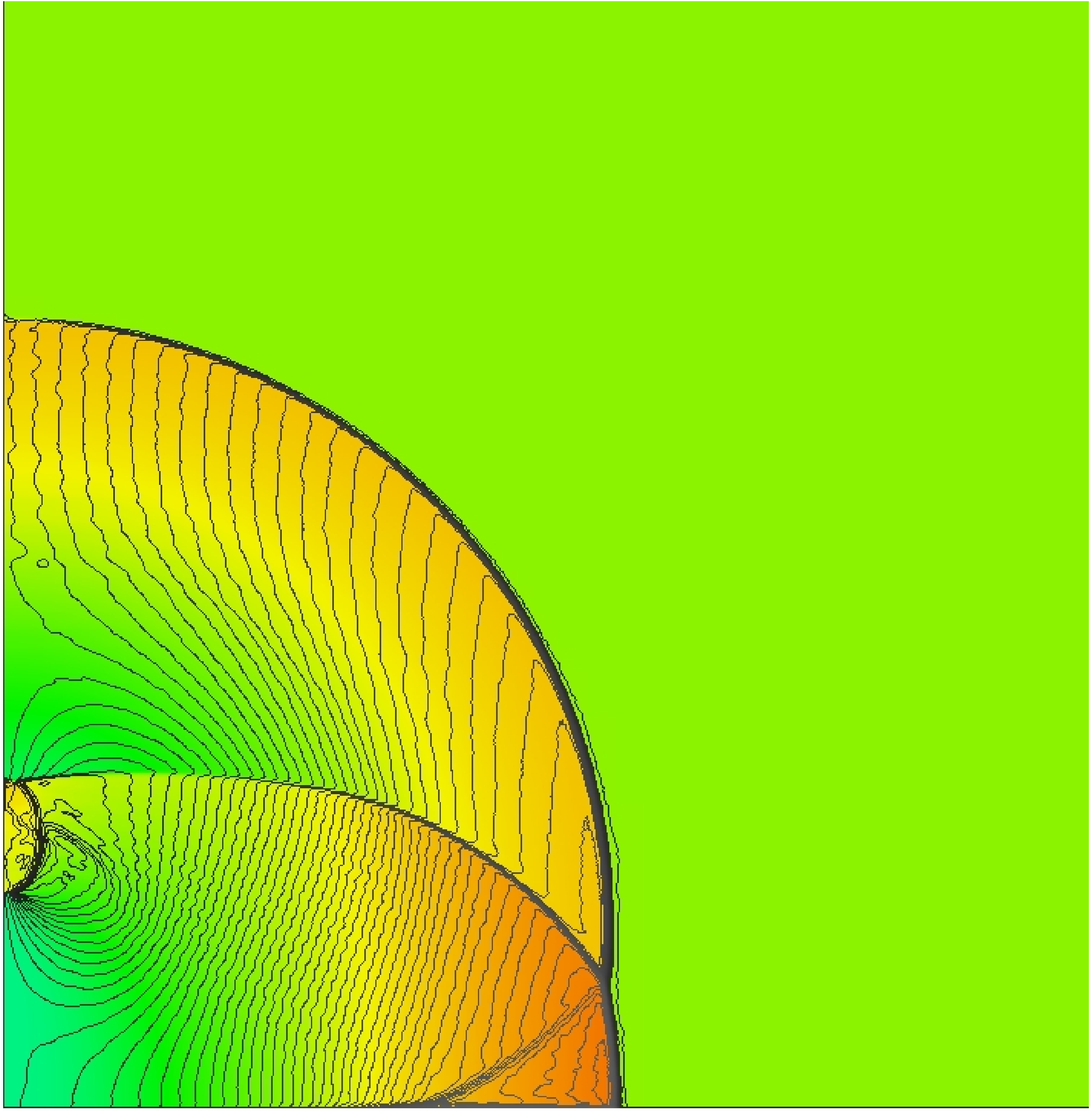


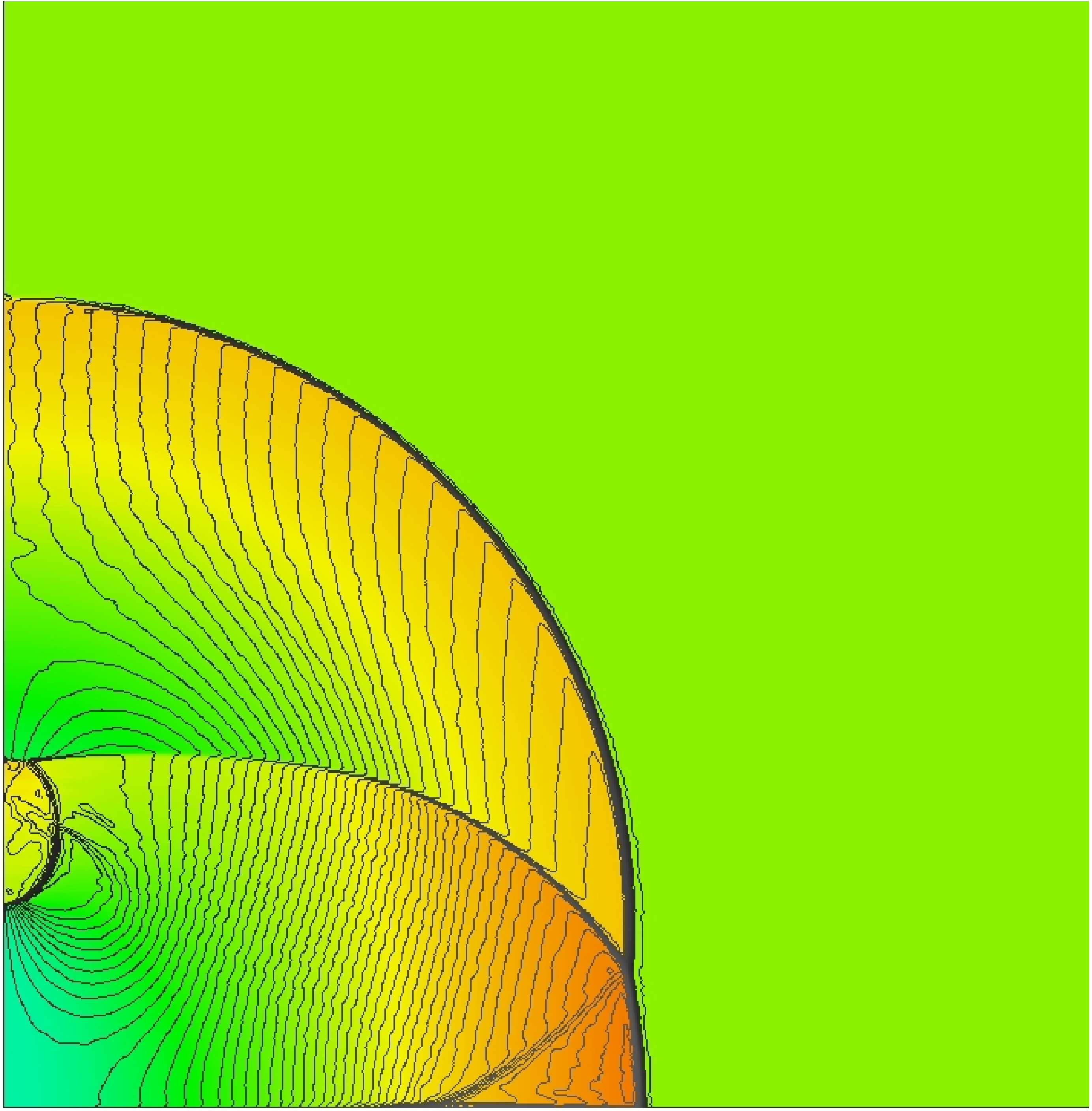


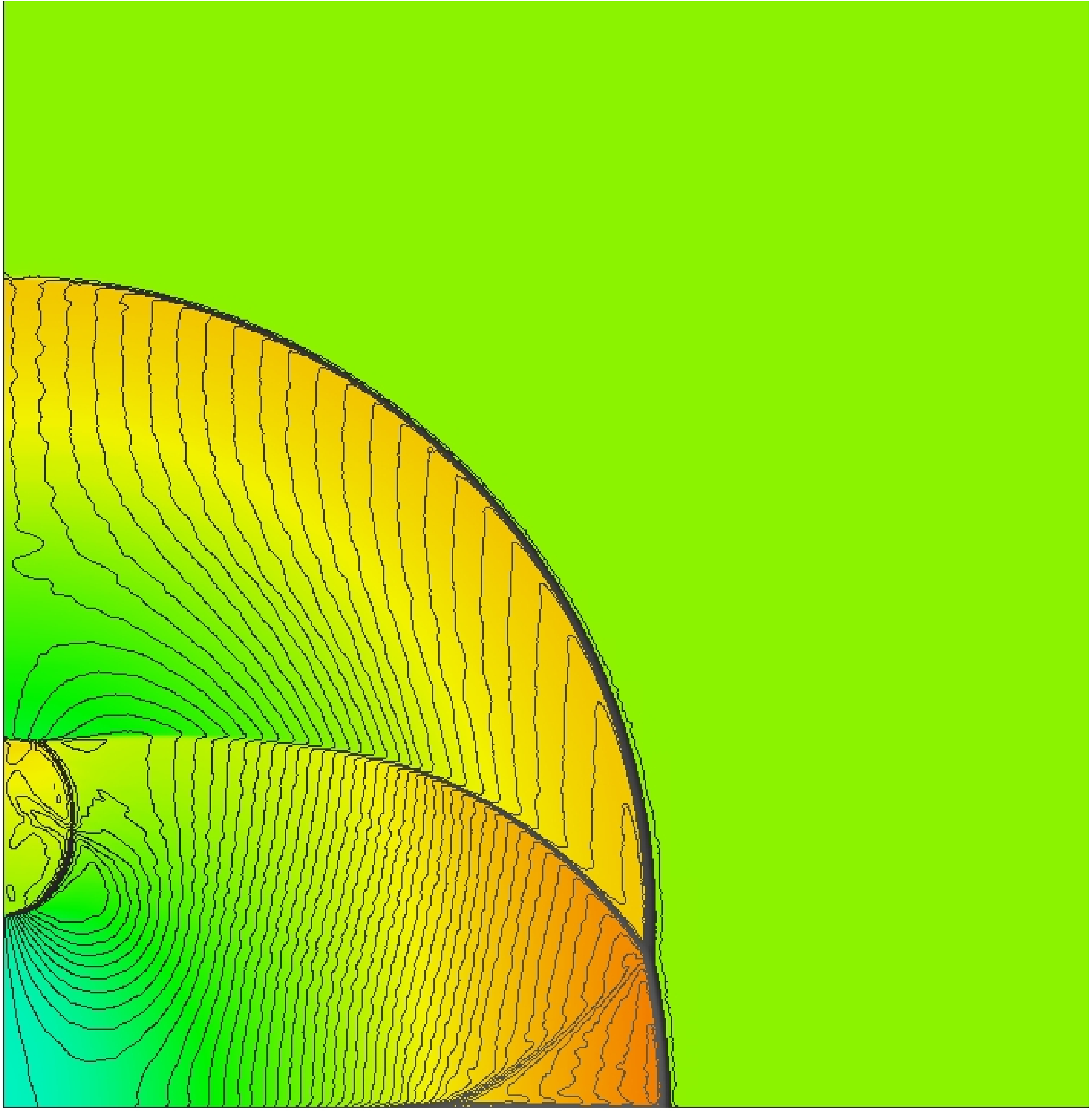


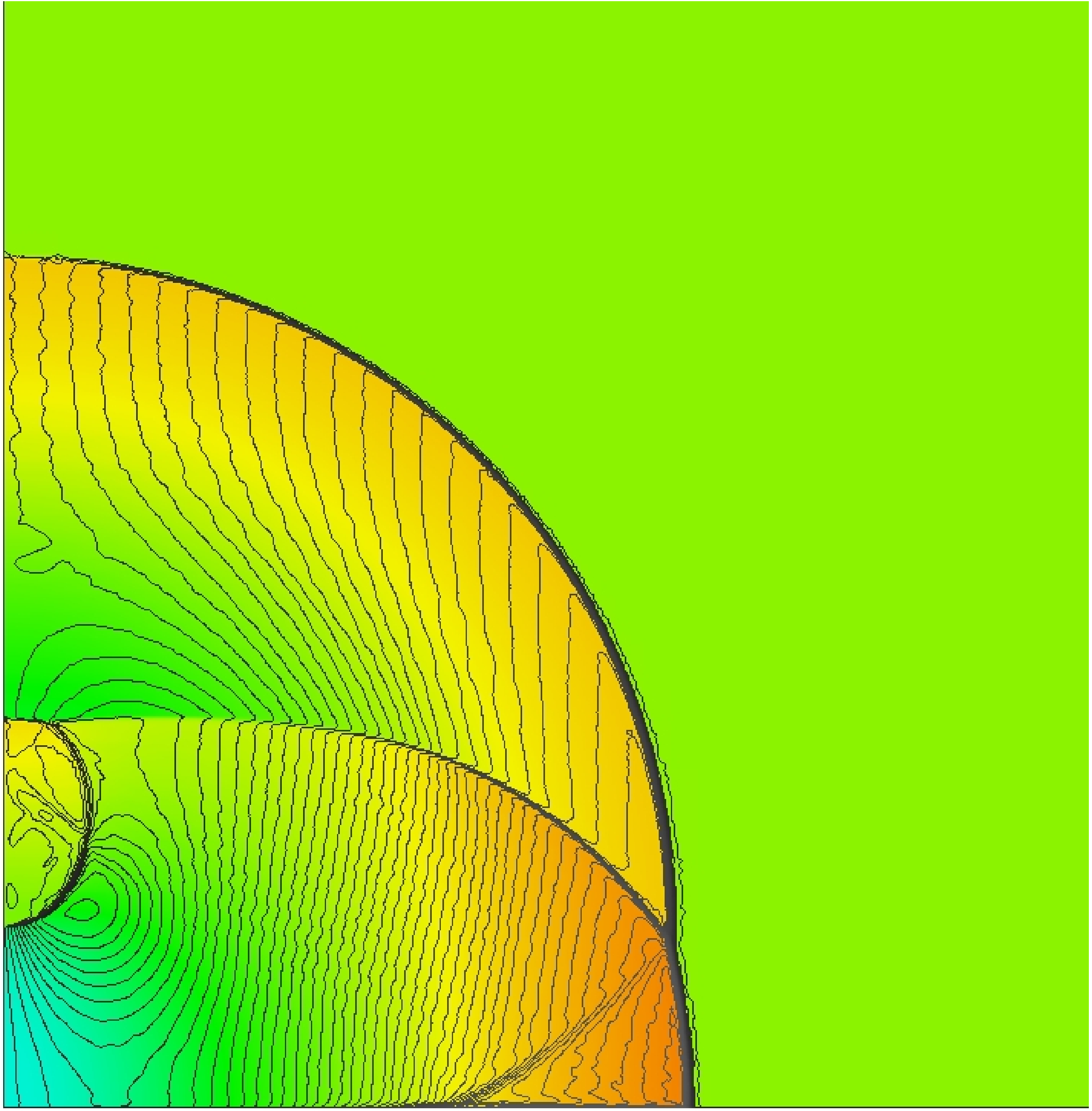


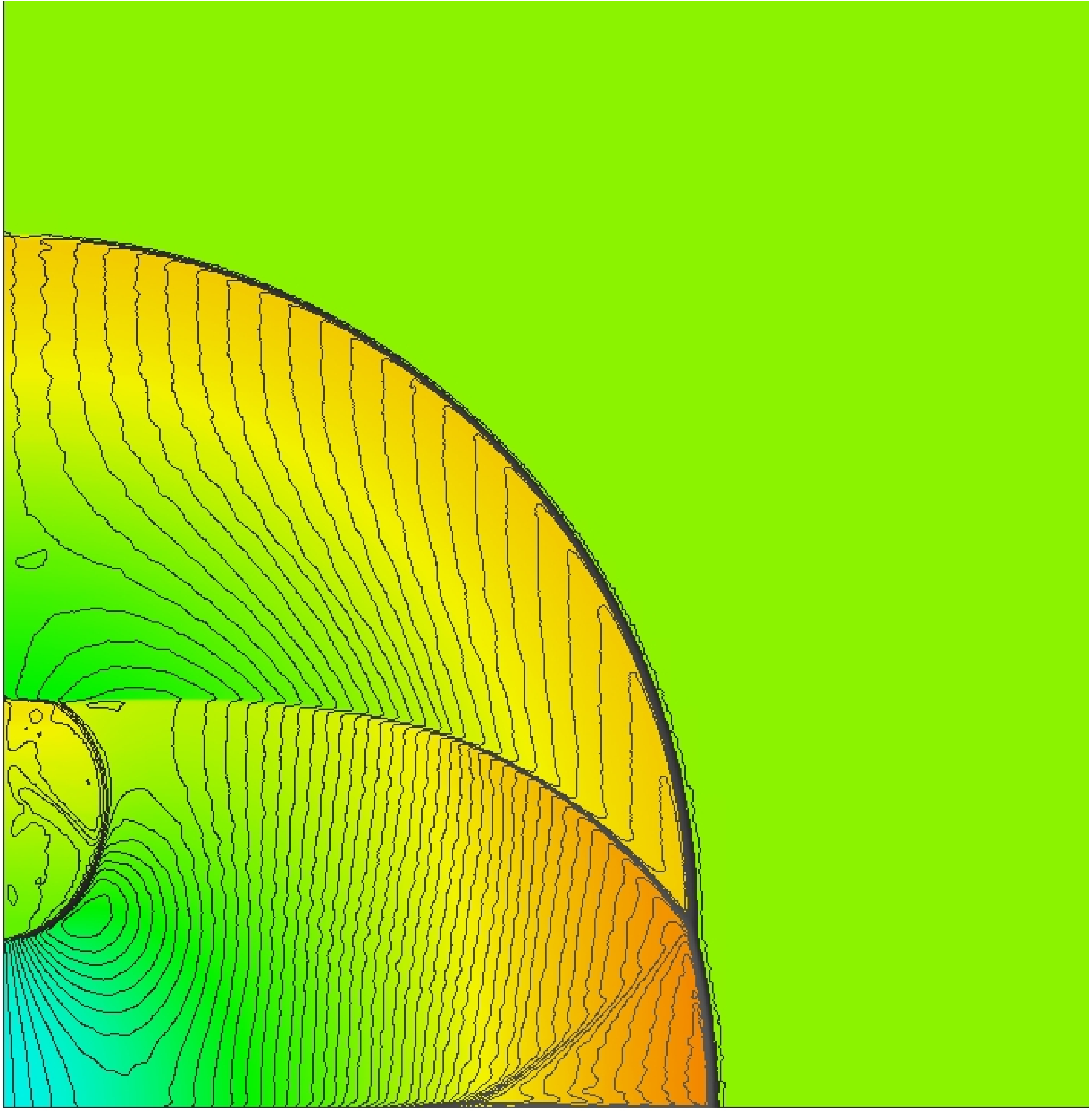


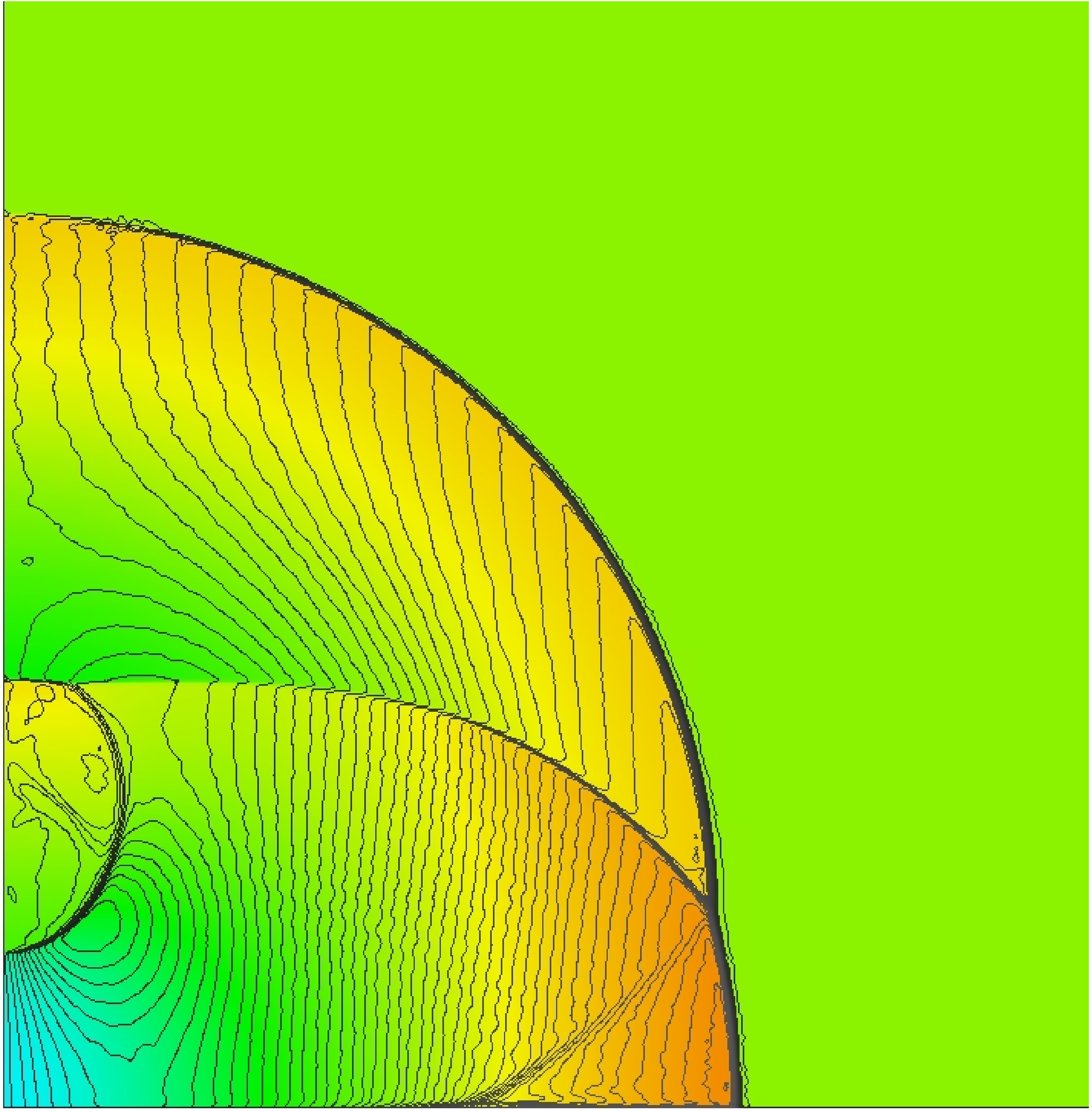


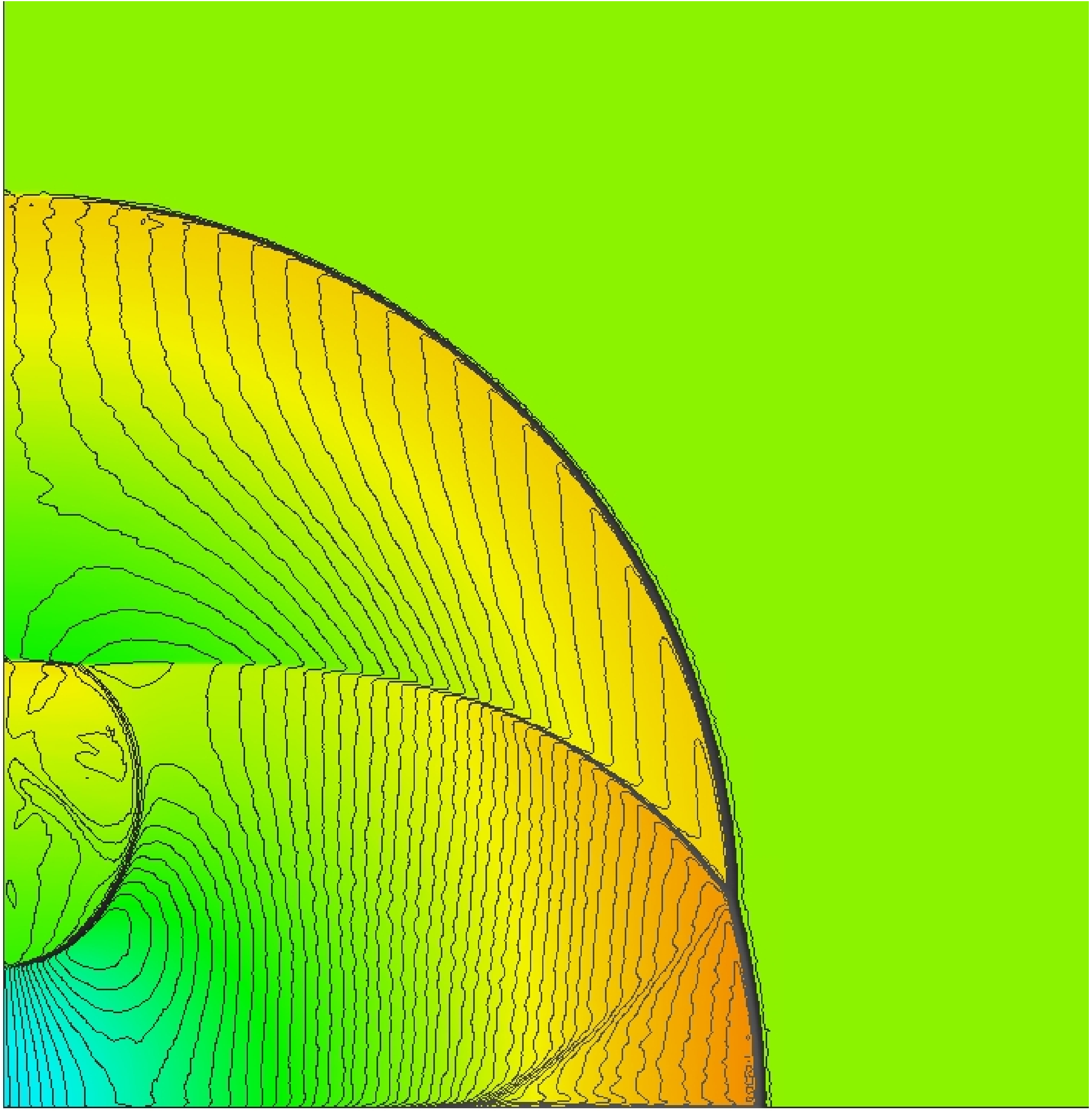


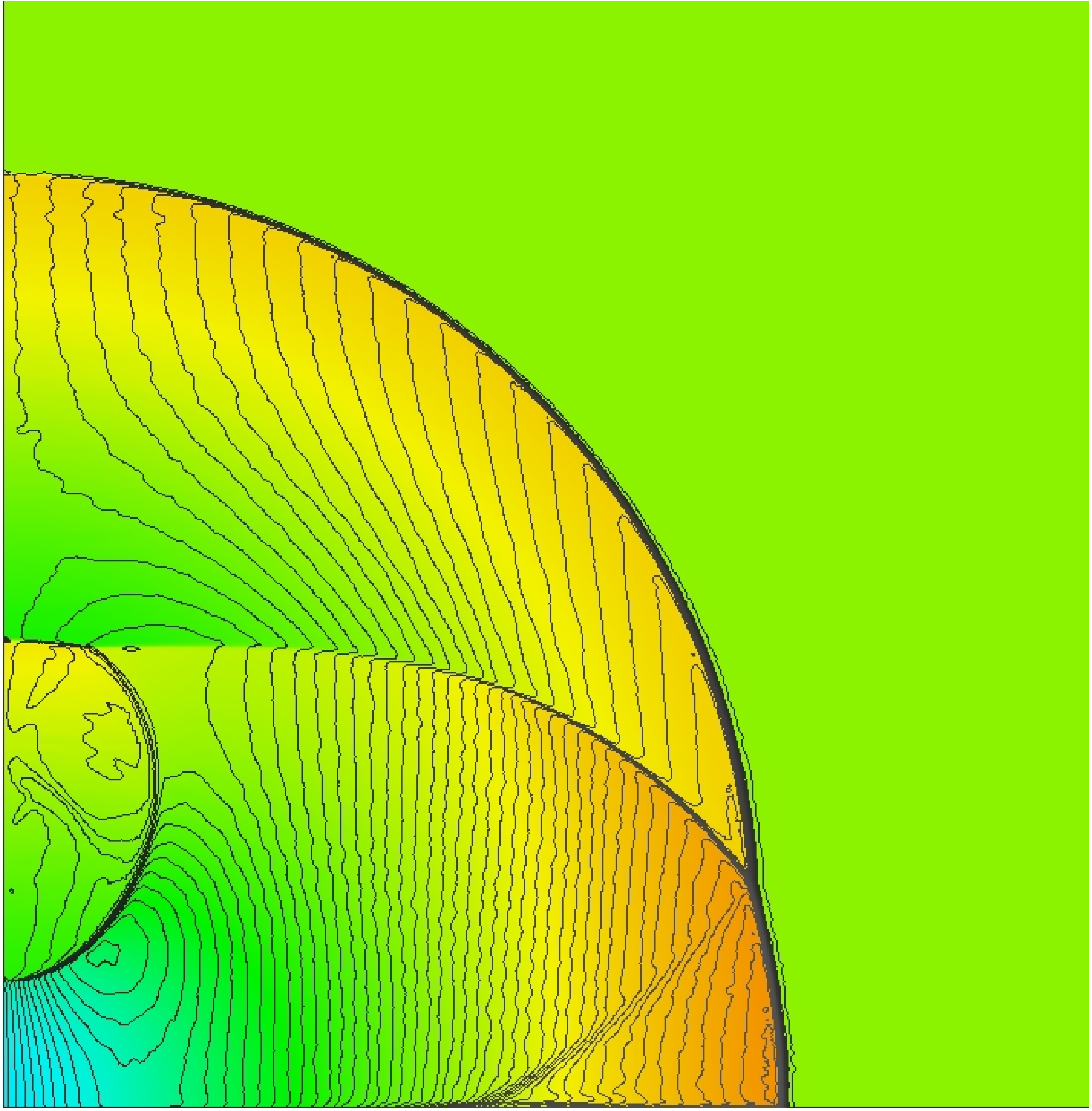


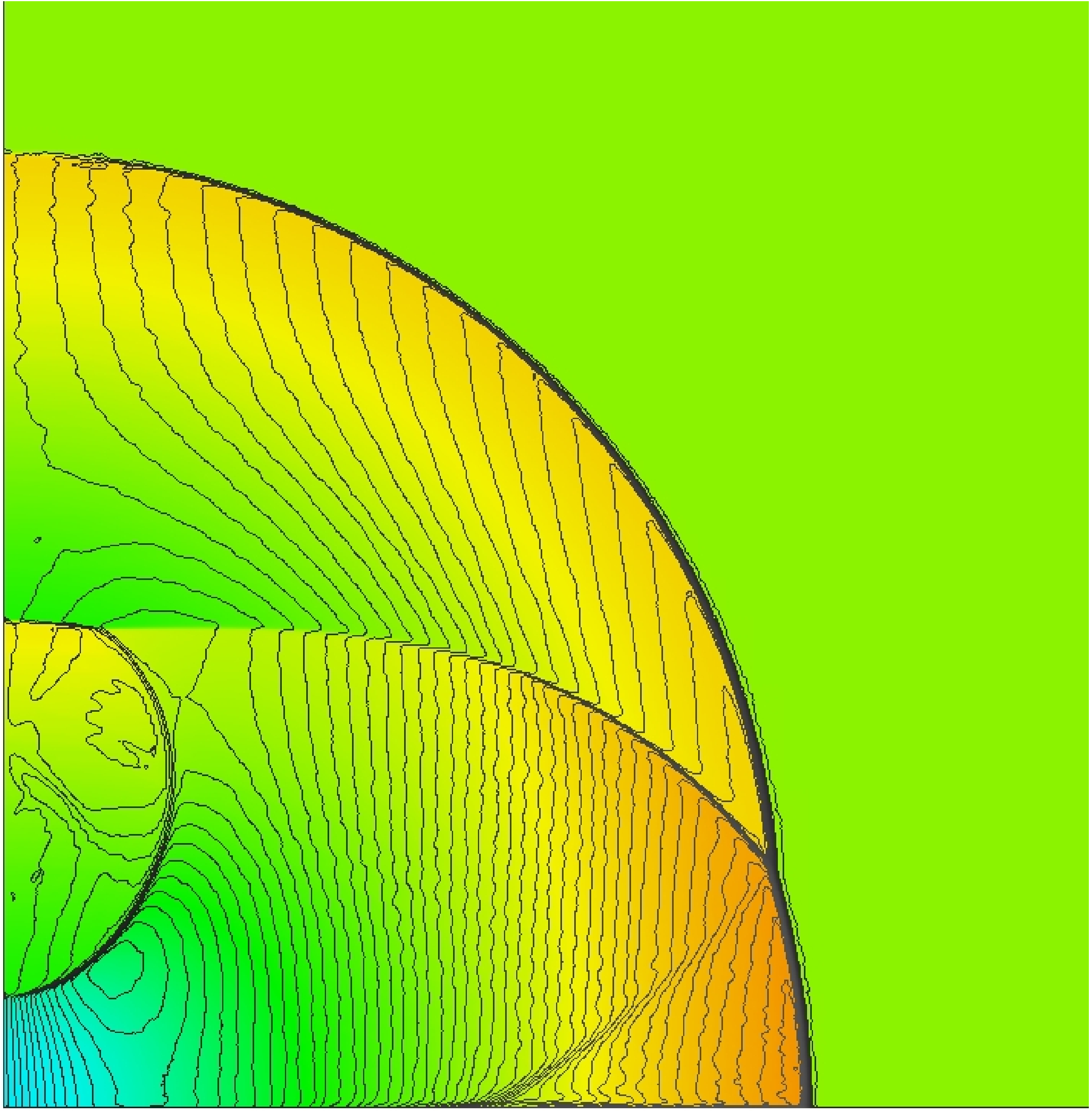


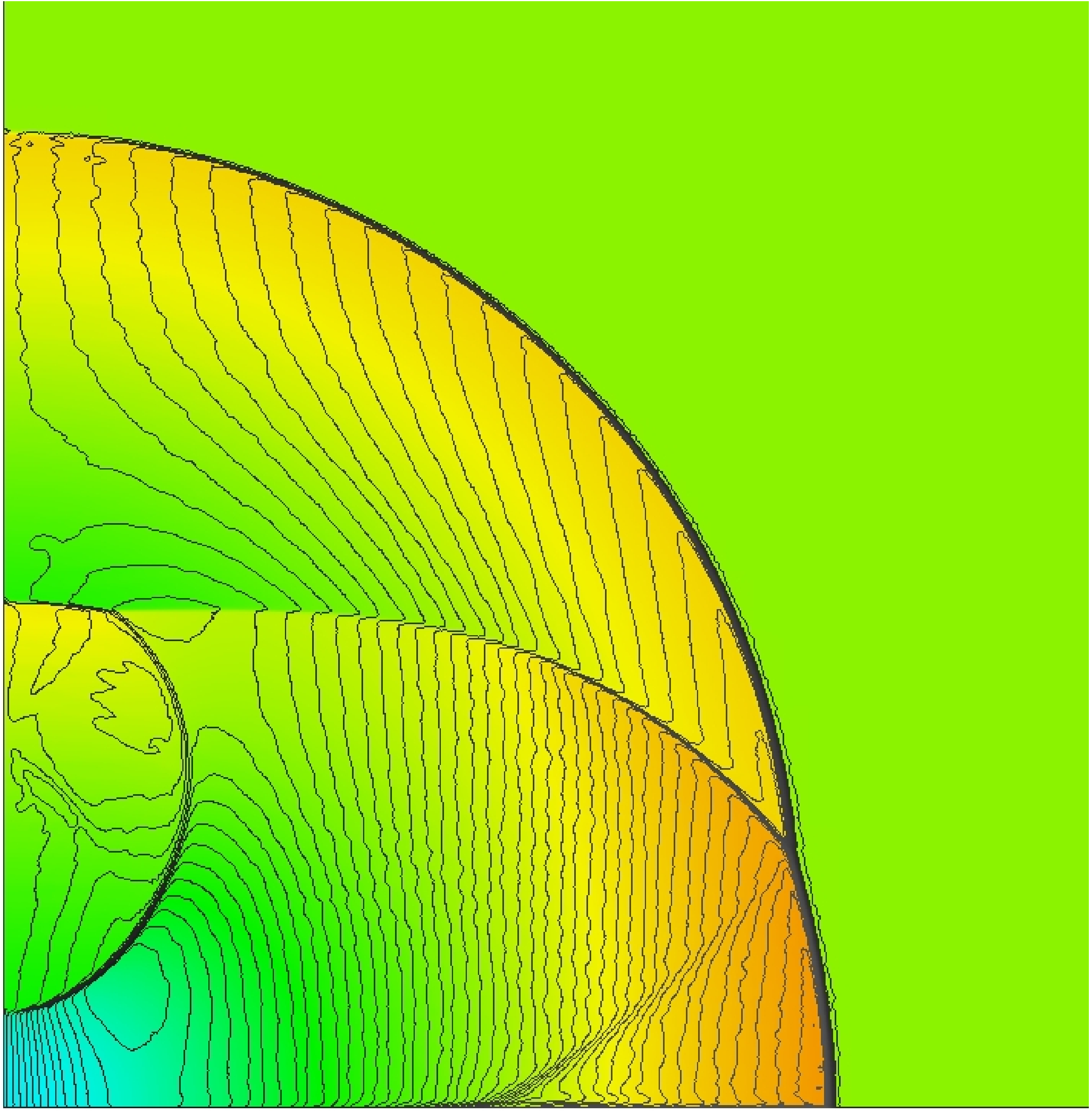


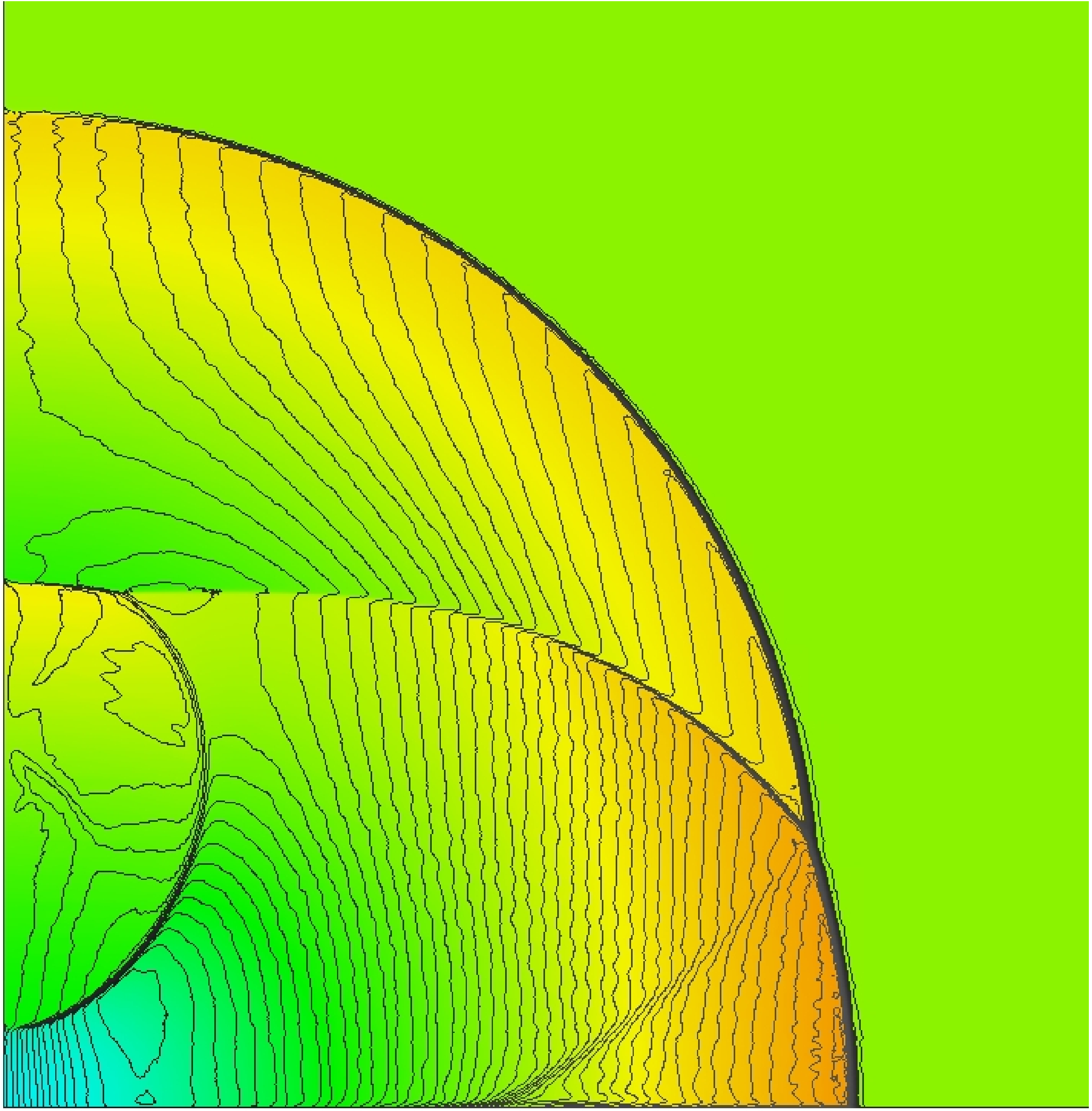


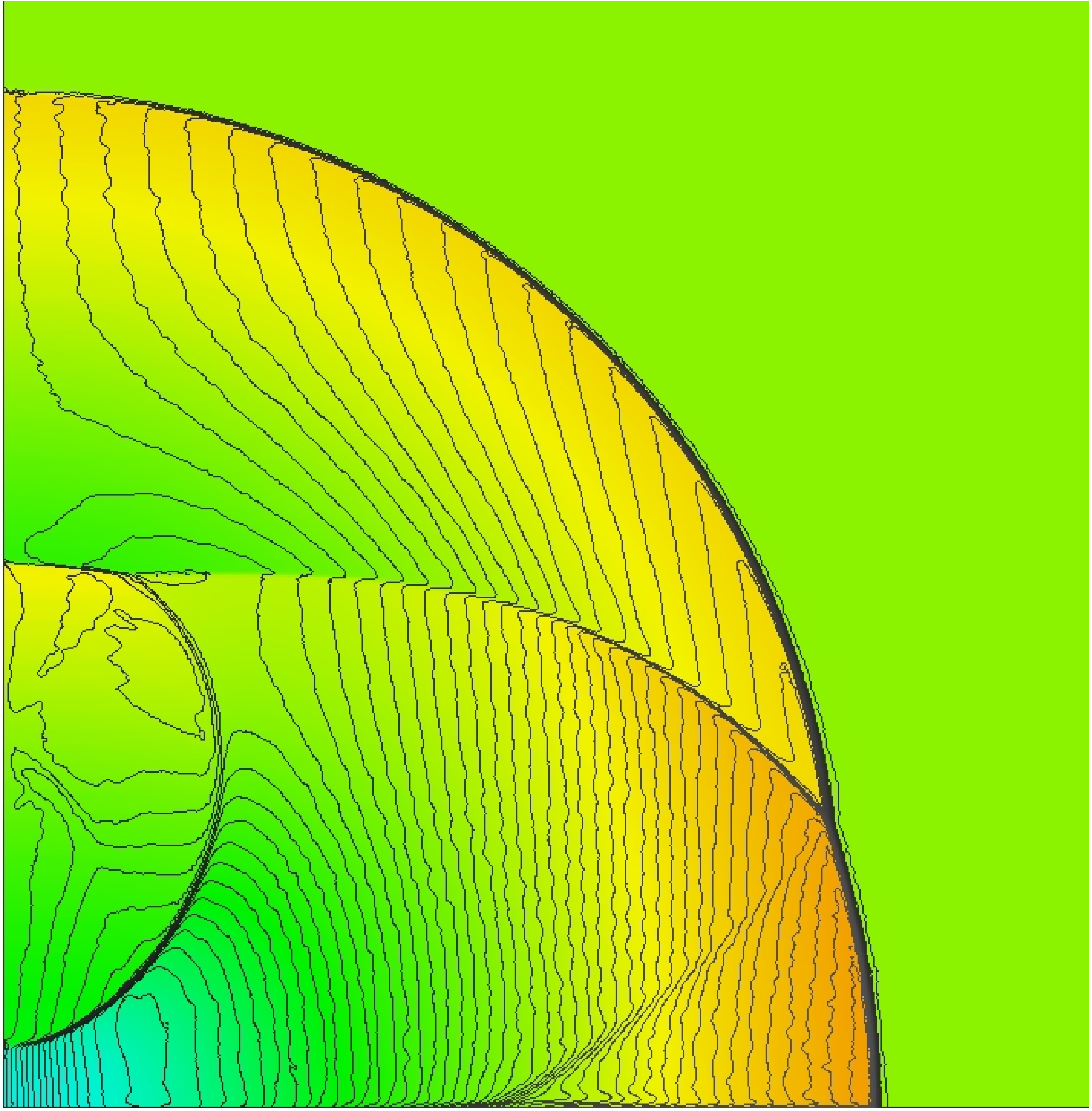


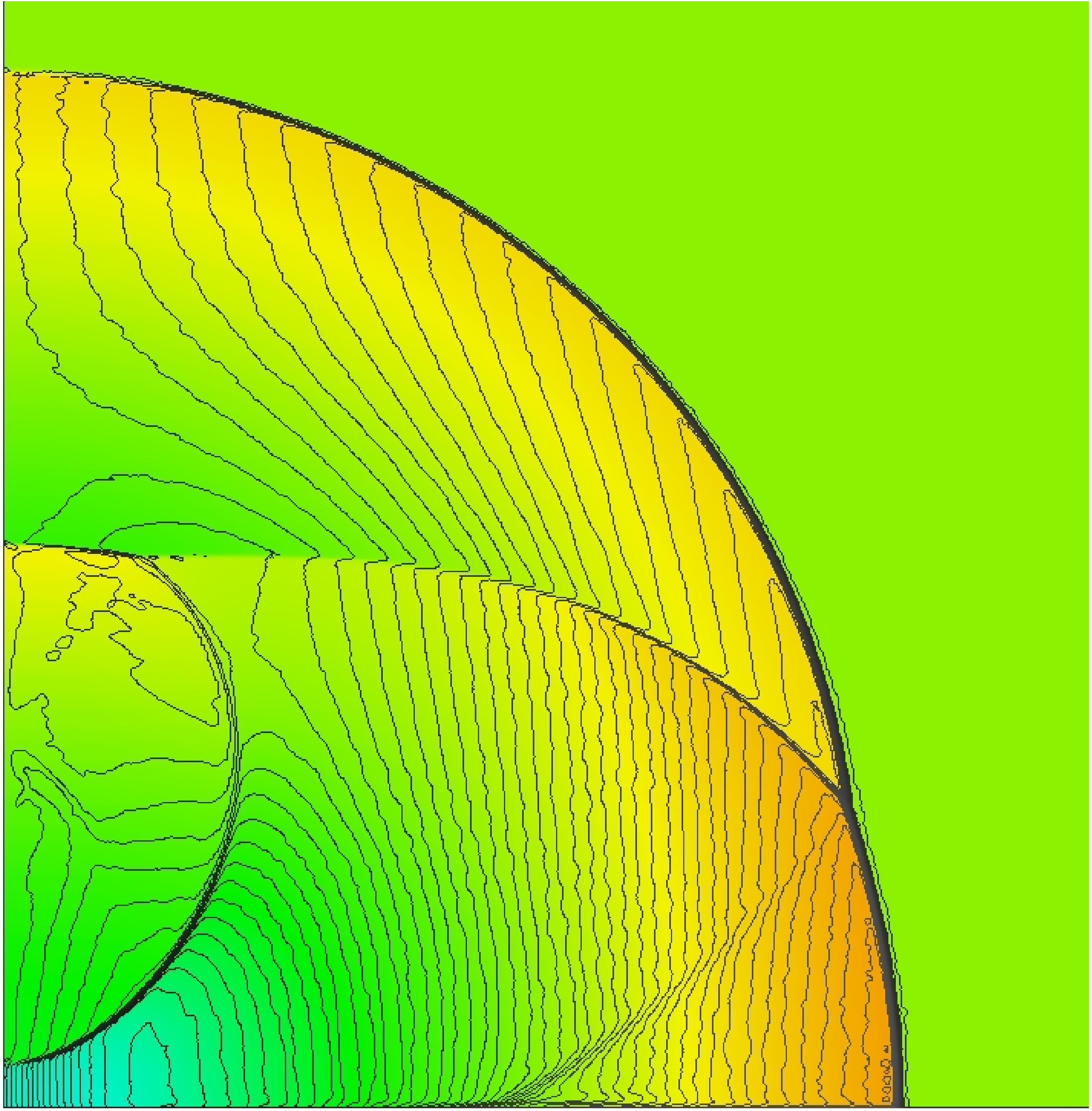


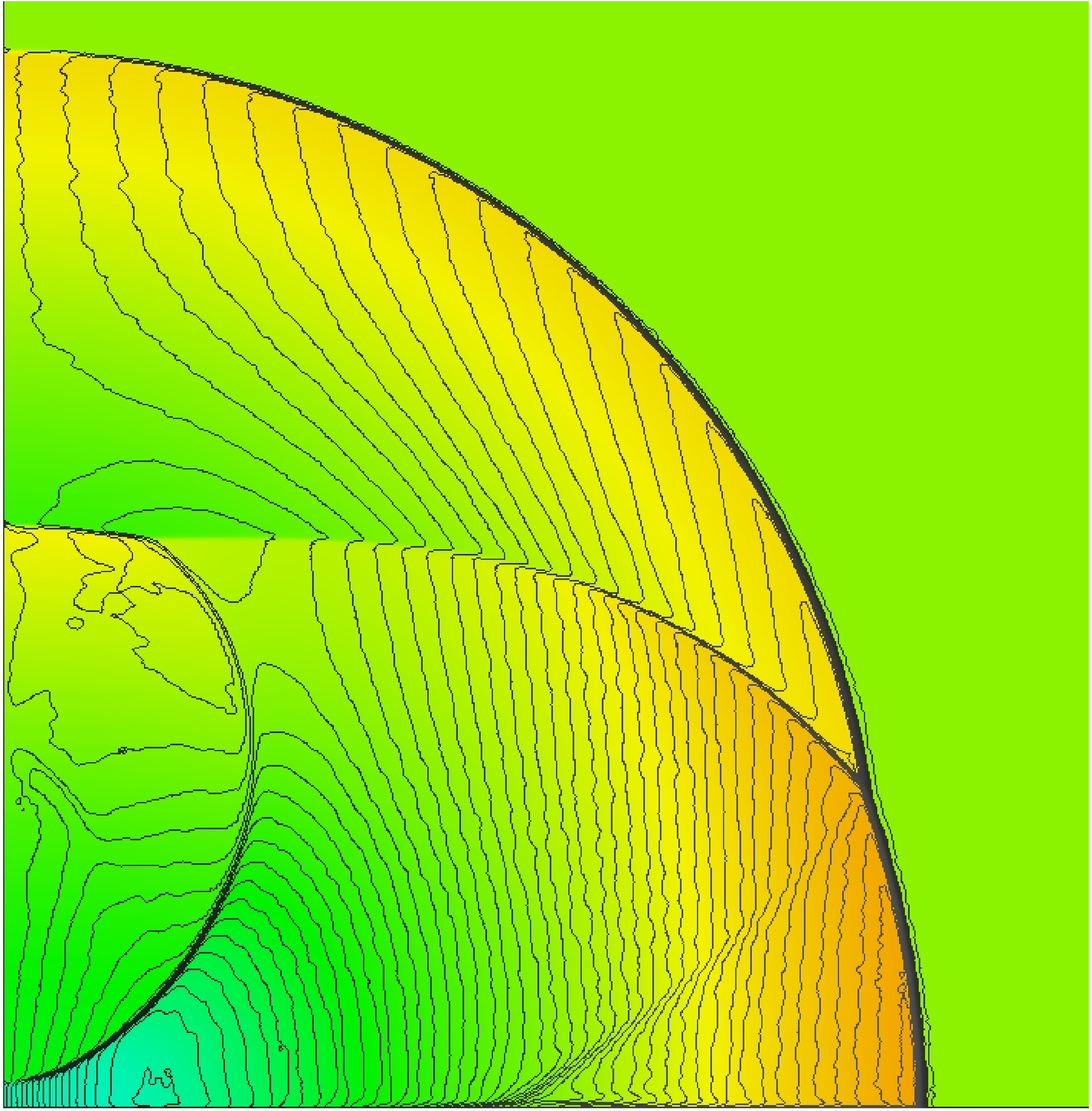






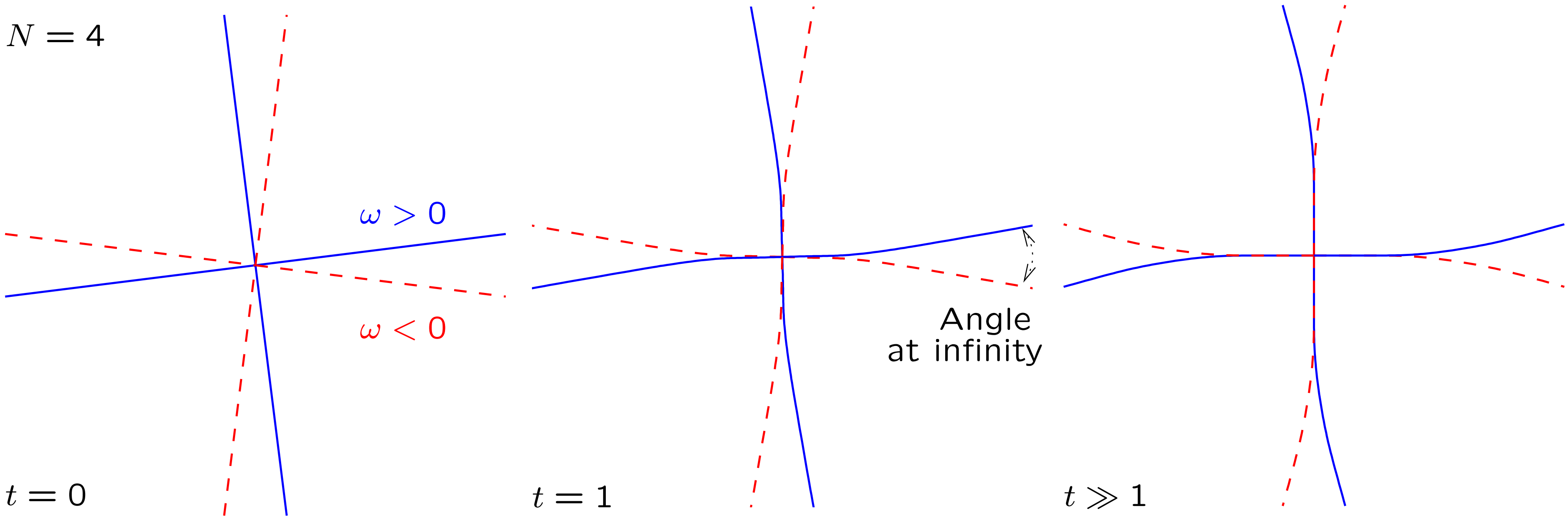






Simple special case

N vortex-sheet pairs, equal angles between center lines



Limit: pair angle at infinity small

Self-similar: μ *similarity exponent*,

$$\boxed{dx \sim t^{\mu}}$$

Self-similarity (Euler):

$$0 = \omega_t + \mathbf{v} \cdot \nabla\omega \quad , \quad 0 = \nabla \cdot \mathbf{v} \quad , \quad \omega = \nabla \times \mathbf{v}$$

Initial data $\omega(t = 0, \mathbf{x}) = |\mathbf{x}|^{-1/\mu}\mathring{\omega}(\measuredangle\mathbf{x})$. Ansatz:

$$\omega(t, \mathbf{x}) = t^{-1}\omega(t^{-\mu}\mathbf{x}) \quad , \quad \mathbf{v}(t, \mathbf{x}) = t^{\mu-1}\mathbf{v}(t^{-\mu}\mathbf{x})$$

Selfsimilar incompressible Euler:

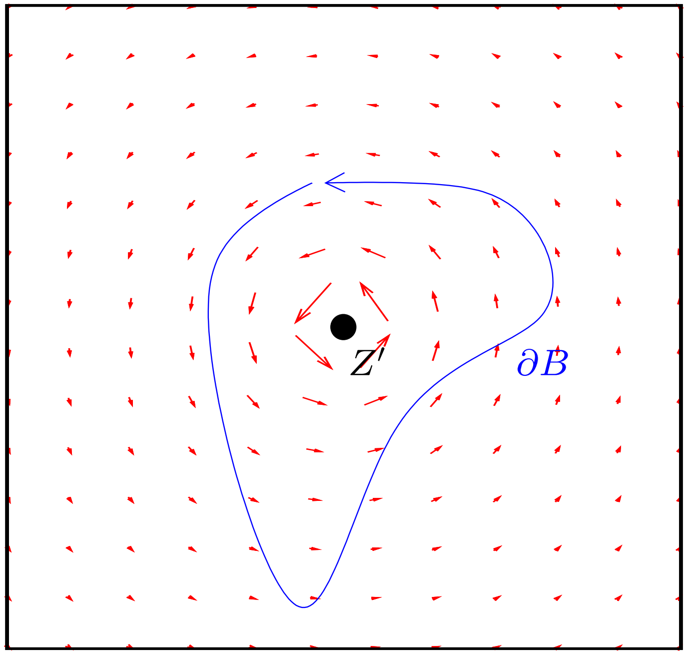
$$0 = (\underbrace{\mathbf{v} - \mu\mathbf{x}}_{=\mathbf{q}}) \cdot \nabla\omega - \omega = \nabla \cdot \Big((\underbrace{\mathbf{v} - \mu\mathbf{x}}_{=\mathbf{q}})\omega\Big) + (2\mu - 1)\omega$$

$\mathbf{q}$ *pseudo-velocity*: along it ω *smoother* than transversally

$$0 = \nabla \cdot \mathbf{v} \quad \Leftrightarrow \quad -2\mu = \nabla \cdot \mathbf{q}$$

Biot-Savart law

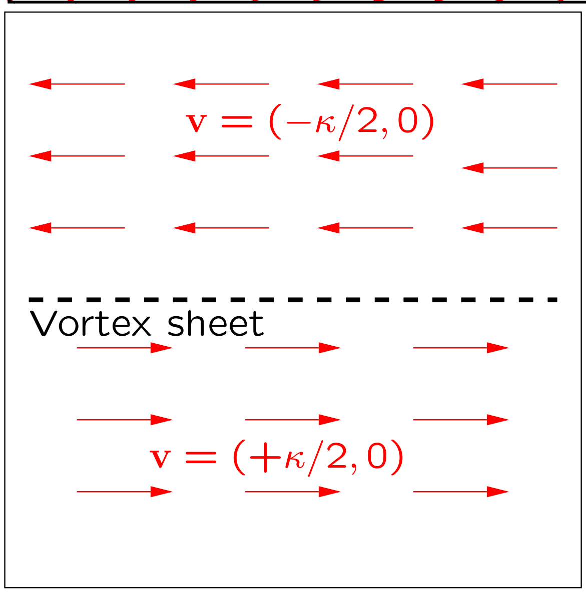
Circulation: $\Gamma(B) = \oint_{\partial B} \mathbf{v} \cdot d\mathbf{x} = \int_B \omega \; dx \; dy$



$Z = x + iy, \; V = v^x - iv^y$

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \nabla \times \mathbf{v} = \omega \end{cases} \Rightarrow V(Z) = \frac{1}{2\pi i} \int \frac{\overbrace{\omega(Z')dx \; dy}^{d\Gamma(Z')}}{Z - Z'}$$

Point vortex: $V(Z) = \dfrac{\Gamma}{2\pi i(Z - Z')} \sim \dfrac{1}{r}$



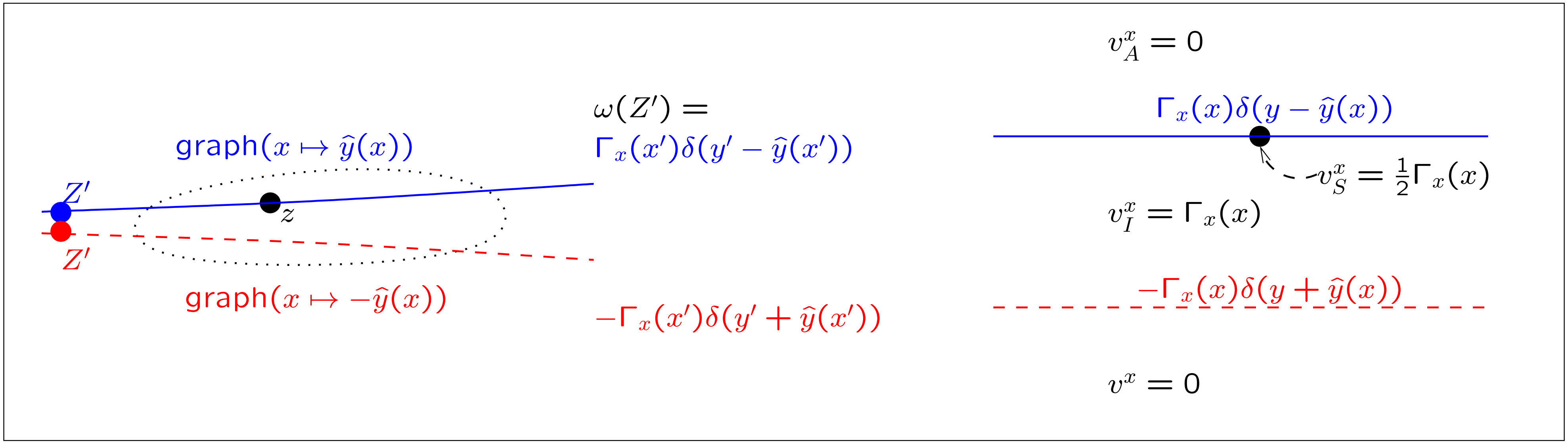
Straight uniform vortex sheet:
$\leftarrow$ infinitesimal point vortices on line,
equal spacing dx, equal $d\Gamma = \kappa \; dx$

$$V(Z) = \begin{cases} -\kappa/2, & y = \text{Im}\, Z > 0 \\ +\kappa/2, & \text{Im}\, Z < 0 \end{cases}$$

Cusp approximation: horizontal part

$\Gamma(x) = \int_C \omega(x)dx$, C ccw loop enclosing upper sheet from 0 to x

$$V(Z) = \frac{1}{2\pi i}\int \frac{\Gamma_x(x')dx'}{Z - Z'}$$



Distant blue points have nearby red point mirror image with equal ω of *opposite*-sign $\to$ strong cancellation

Nearby parts dominate V integral

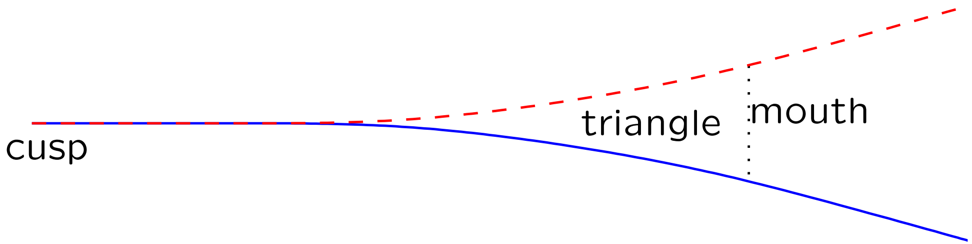
Horizontal velocity $v^x = \mathrm{Re}\, V$:

$$v^x(x,y) \approx v^x_I(x) = \Gamma_x(x) \qquad \text{inside}$$

$$v^x(x,y) \approx v^x_S(x) = \frac{1}{2}\Gamma_x(x) \qquad \text{on (upper) sheet}$$

Mass conservation

$$-2\mu = \nabla \cdot \mathbf{q} \quad , \quad \mathbf{q} \cdot \mathbf{n} = 0 \quad \text{at sheet} \quad (\mathbf{q} = \mathbf{v} - \mu \mathbf{x})$$



Integrate over any "cusp triangle":

$$0 > -2\mu \cdot \text{triangle area} = \int_{\text{upper,lower sides}} \underbrace{\mathbf{q} \cdot \mathbf{n}}_{=0} \, ds + \int_{\text{mouth}} q^x(x, y) dy$$

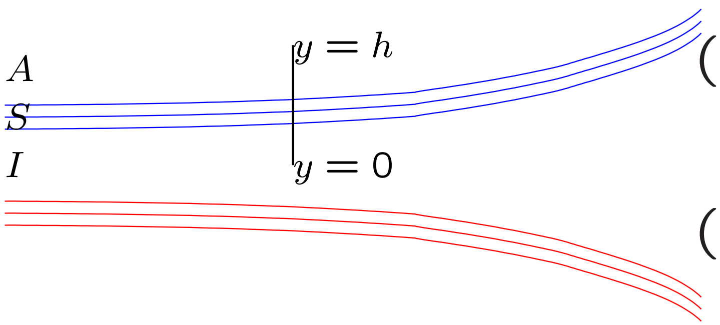
Approximation has $v^x(x, y) \approx v^x(x)$ and hence

$$q^x(x, y) \approx q^x(x) < 0$$

$\rightsquigarrow$ only consider solutions with

$$q_I^x < 0 \quad , \quad q_I^x \to 0 \quad \text{as } x \searrow 0$$

Horizontal velocity modelling Selfsimilar vorticity equation:



$$(1-2\mu)\omega = \Big((v^x-\mu x)\omega\Big)_x + \Big((v^y-\mu y)\omega\Big)_y$$

$$(1-2\mu)\int_0^h \omega dy = \Big(\int_0^h (v^x-\mu x)\omega dy\Big)_x + \underbrace{\Big[(v^y-\mu y)\omega\Big]_0^h}_{=0}$$

For small pair angle $v^y, v^x_A, \hat{y} \to 0$, but v^x not. So:

$$\omega = v^y_x - v^x_y \approx -v^x_y \quad \Rightarrow \quad (1-2\mu)\int_0^h v^x_y dy = \Big(\int_0^h (v^x-\mu x) v^x_y dy\Big)_x$$

$$= \Big(\int_0^h \Big(\frac{(v^x)^2}{2}\Big)_y - \mu (x v^x)_y dy\Big)_x$$

$$(1-2\mu)[v^x]^h_{y=0} = \Big([\frac{1}{2}(v^x)^2 - \mu x v^x]^h_{y=0}\Big)_x$$

$$v^x_A \approx 0 \quad \overset{\text{no } h}{\underset{\text{only } 0}{\Rightarrow}} \quad (1-2\mu)v^x_I = \Big(\frac{(v^x_I)^2}{2} - \mu x v^x_I\Big)_x$$

ODE for velocity $v^x = v^x_I(x)$ between sheets!

Horizontal velocity approximation

$$(1-2\mu)v_I^x = \Big(\frac{(v_I^x)^2}{2} - \mu x v_I^x\Big)_x \quad \Rightarrow \quad (1-\mu)v_I^x = (v_I^x - \mu x)\partial_x v_I^x$$

$$v^x(x) = xu(x) \quad \Rightarrow \quad -\frac{\partial u}{\partial \log x} = \frac{u(u-1)}{u-\mu}$$

- Conservation: $q_I^x \nearrow 0$ at cusp $x \searrow 0$, so $u = v_I^x/x = o(\frac{1}{x})$.
- Need $q_I^x = v_I^x - \mu x < 0$ everywhere, so $u < \mu$.
- At $x \to \infty$ need $v^x = o(x)$ ($\leftarrow$ "uniqueness"), so $u = o(1)$.
- Analysis: get $O(1/x)$ blowup as $x \searrow 0$, or $u \to \mu$ and hence $q^x \to 0$ at finite $x > 0$ (no use), unless $u \to 0$ or $u \to 1$.
- $u = 0$ unstable for *any* $\mu > \frac{1}{2}$.
- $u = 1$: for $\mu < 1$: unstable.

For $\mu = 1$: $u = 1$ absent (cancels).

For $\mu > 1$: $u = 1$ asy. stable $\rightsquigarrow$ nontrivial $v^x = x + o(x)$ solutions.

Conclusion: only for $\mu > 1$ have useful solution:

$$\boxed{v_I^x(x) = x + \ldots \quad , \quad v_S^x(x) = \frac{1}{2}x + \ldots \quad , \quad q_S^x(x) = (\frac{1}{2} - \mu)x + \ldots}$$

Vertical velocity approximation 1: just zero

Assume v^y inside and outside is tiny, can be neglected:

$$v_S^y \approx 0$$

ODE: $\mathbf{q}_S$ tangential to sheet graph $\widehat{y}$, so

$$\widehat{y}_x = \frac{q_S^y}{q_S^x} = \frac{v_S^y - \mu\widehat{y}}{v_S^x - \mu x} \approx \frac{0 - \mu\widehat{y}}{(\frac{1}{2} - \mu)x} \qquad \Rightarrow \qquad (\frac{1}{2} - \mu)x\widehat{y}_x \approx -\mu\widehat{y}$$

Solution:

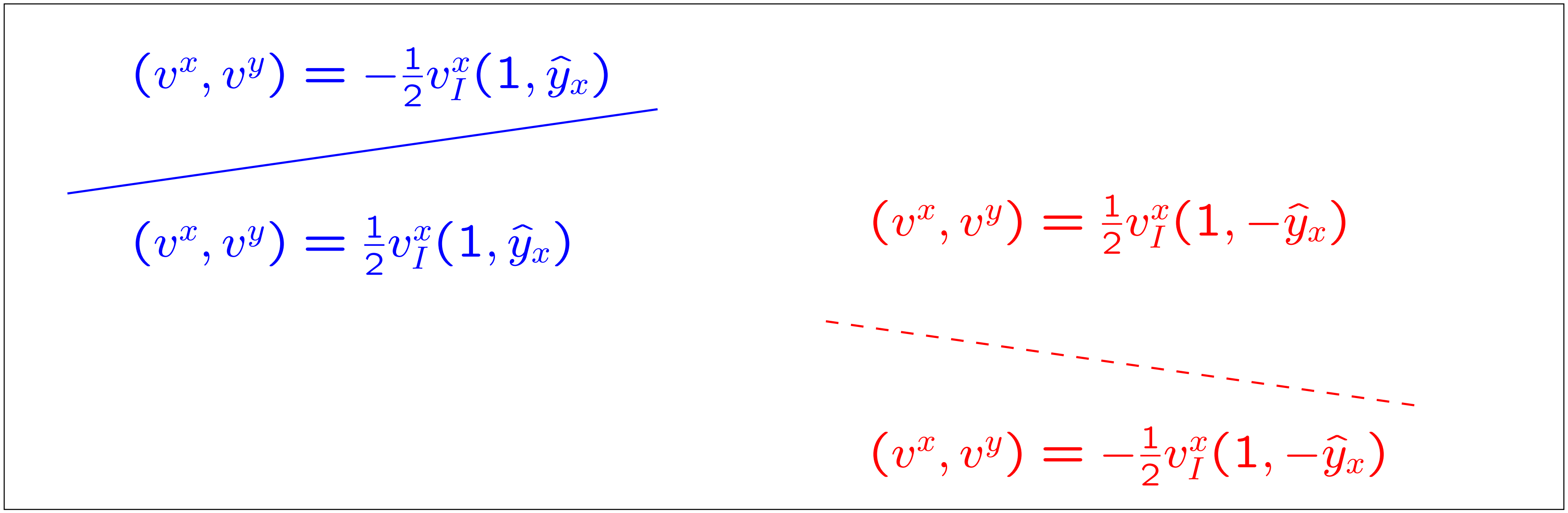
$$\widehat{y} = Cx^{\mu/(\mu - \frac{1}{2})} \qquad \text{for } x \approx 0$$

Cusp exponent:

$$\boxed{\frac{\mu}{\mu - \frac{1}{2}}}$$

v^y approximation 2: non-horizontal sheet

Sheet tangent: $(1, \widehat{y}_x)/\sqrt{1+\widehat{y}_x^2}$. To first order in $\widehat{y}$: tangent $(1, \widehat{y}_x)$



$$v_A^y \approx -v_I^x \widehat{y}_x \quad , \quad v_S^y \approx -\frac{1}{2} v_I^x \widehat{y}_x \quad , \quad v_I^y \approx 0$$

ODE:

$$\widehat{y}_x = \frac{v_S^y - \mu \widehat{y}}{v_S^x - \mu x} \approx \frac{-\frac{1}{2} x \widehat{y}_x - \mu \widehat{y}}{(\frac{1}{2} - \mu) x} \quad \Rightarrow \quad (1-\mu) x \widehat{y}_x \approx -\mu \widehat{y}$$

Cusp exponent:

$$\boxed{\frac{\mu}{\mu - 1}}$$

v^y approximation 3: mass conservation

Inside $v_I^x = x + ...$, i.e. $v_x^x = 1 + ...$, so by $v_x^x + v_y^y = 0$ get $v_y^y = -1 + ...$, hence using $v^y = 0$ at $y = 0$ integration to $y = \widehat{y}$ yields

$$v^y \approx -\widehat{y} \qquad \text{on lower side of sheet}$$

Since $\mathbf{v}$ *jump* to upper side is tangential,

$$v^y \approx -\widehat{y} - v_I^x \widehat{y}_x \approx -\widehat{y} - x\widehat{y}_x \qquad \text{on upper side}$$

and averaging yields

$$v_S^y \approx -\widehat{y} - \frac{1}{2} x\widehat{y}_x \qquad \text{on sheet (p.v.)}$$

ODE

$$\widehat{y}_x = \frac{v_S^y - \mu\widehat{y}}{v_S^x - \mu x} \approx -\widehat{y} - \frac{1}{2} x\widehat{y}_x - \mu\widehat{y}(\frac{1}{2} - \mu)x$$

$$(1-\mu)x\widehat{y}_x = -(1+\mu)\widehat{y}$$

Cusp exponent

$$\boxed{\frac{\mu+1}{\mu-1}}$$

Numerics Use Birkhoff-Rott equation ($\Leftrightarrow$ Euler for smooth sheets):

$$Z = Z(\Gamma) \quad , \quad (1-2\mu)\Gamma Z_\Gamma = V^* - \mu Z \quad , \quad V = \int \frac{d\Gamma'}{Z(\Gamma) - Z(\Gamma')}$$

Derivative: by finite differences

Approximate graph $\hat{y}$ by points, interpolate by polynomial segments

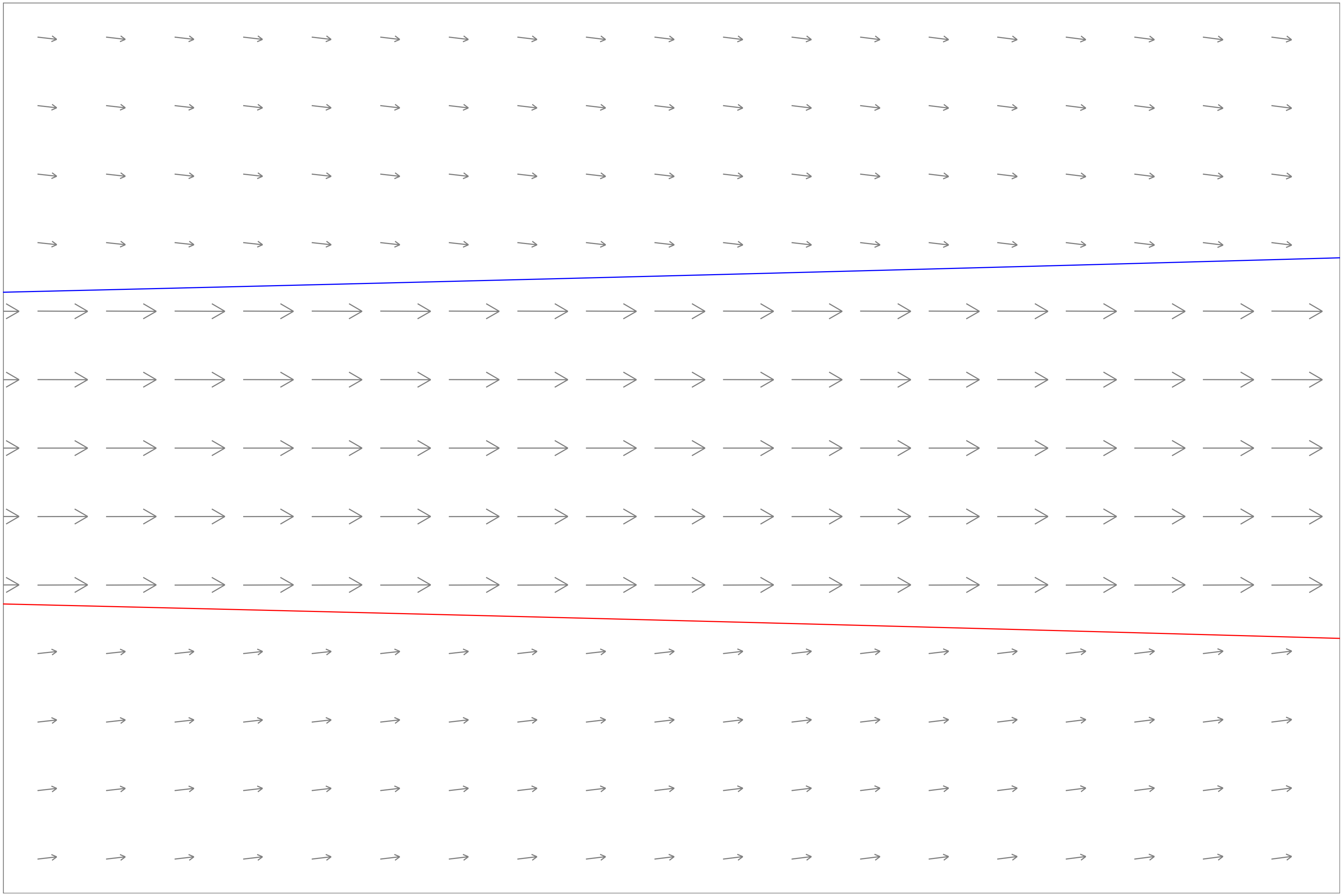
$\rightarrow$ can evaluate Biot-Savart integrals

- Linear interpolation: Convergence to cusp with $\mu/(\mu-1)$ exponent (2nd approx). Not very stable especially as $\mu \searrow 1$.
- Quadratic/cubic/... interpolation: convergence to $(\mu+1)/(\mu-1)$ exponent cusp (3rd approx). More robust if oscillation avoided.
- Linear/higher mixed:

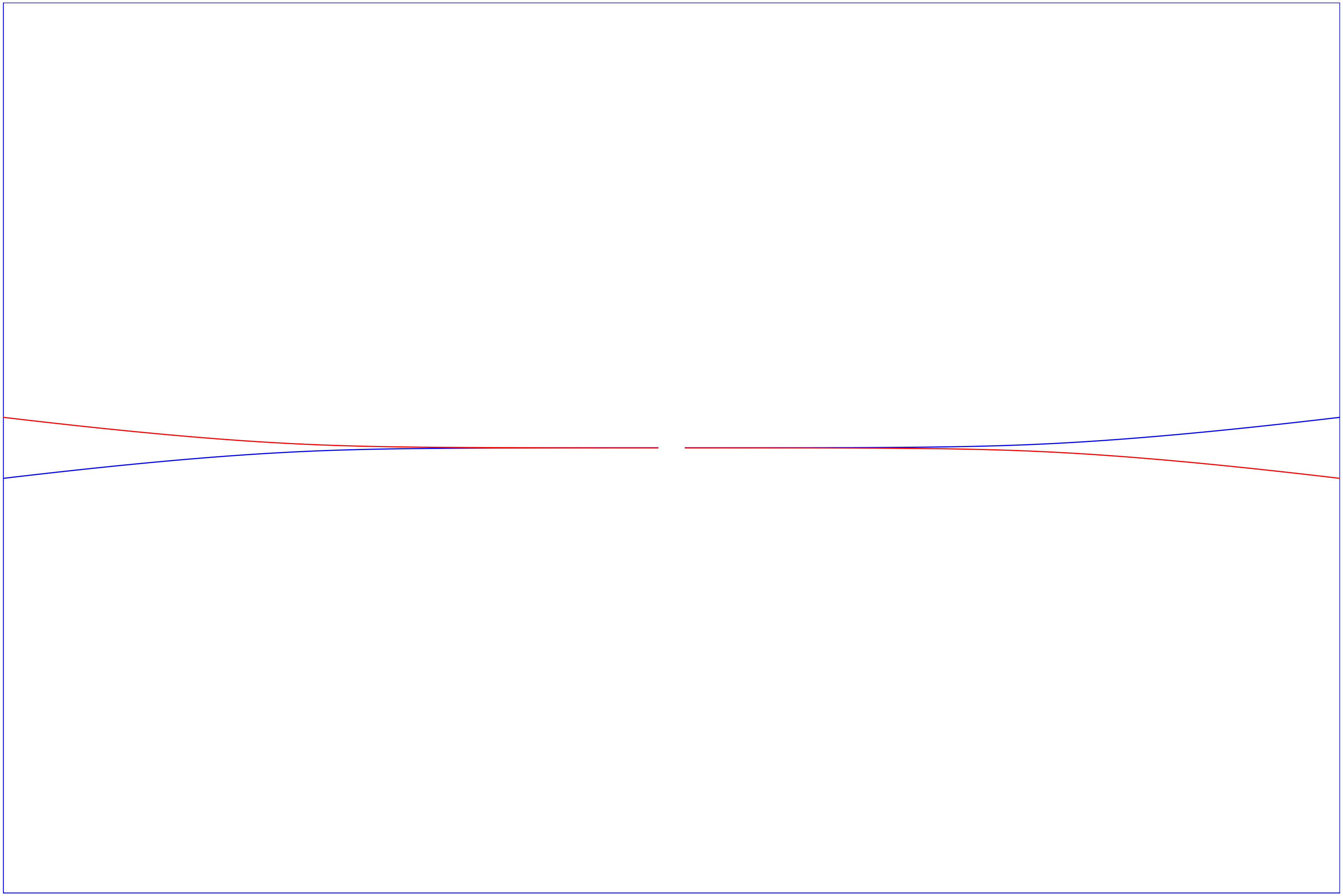
$$V(Z) = \frac{1}{2\pi i} \int \frac{\omega(Z')dx\ dy}{Z - Z'}$$

Enough to use quadratic/higher for *same* z segment and *mirror image*, for rest can use linear

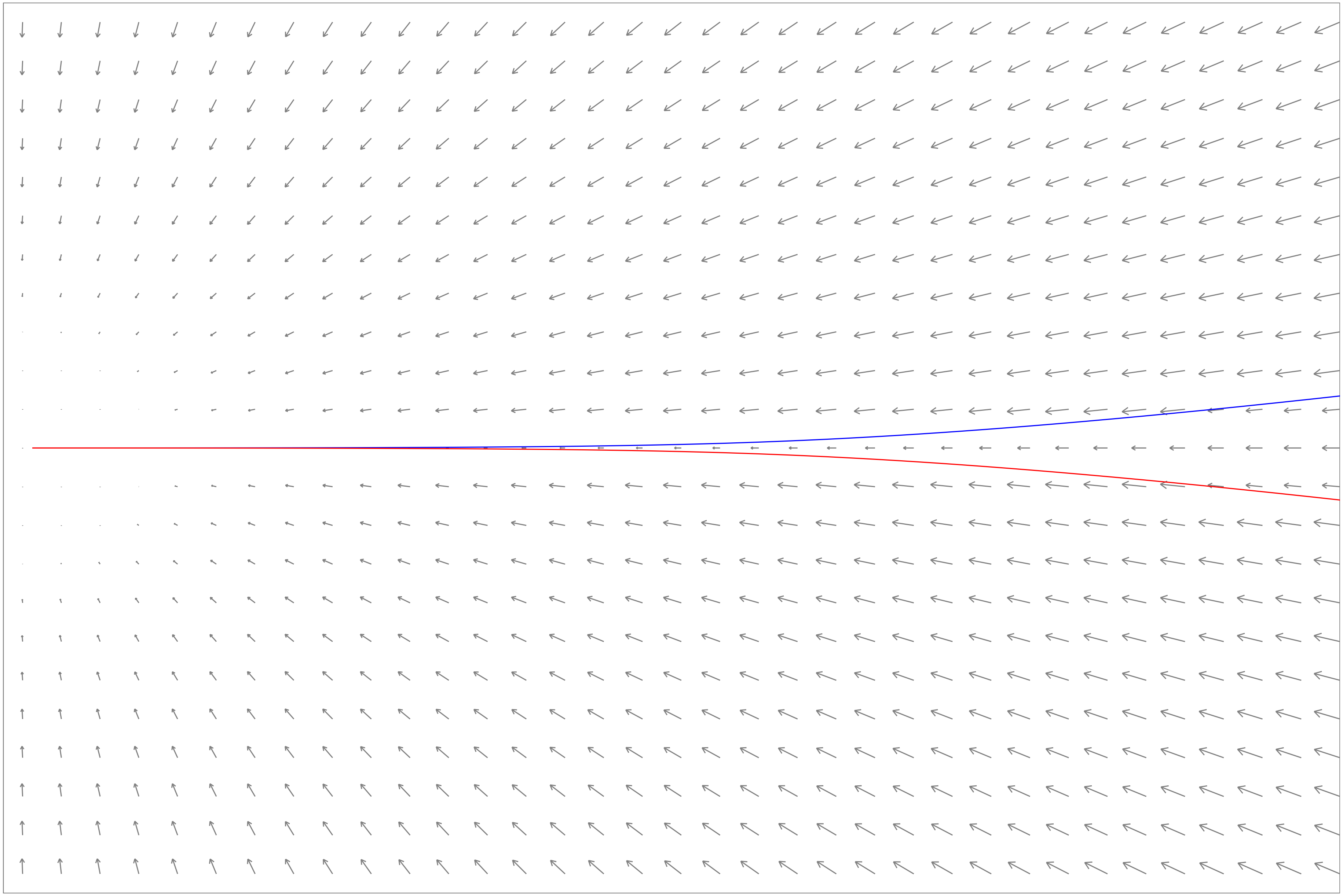
Velocity field (zoom in):



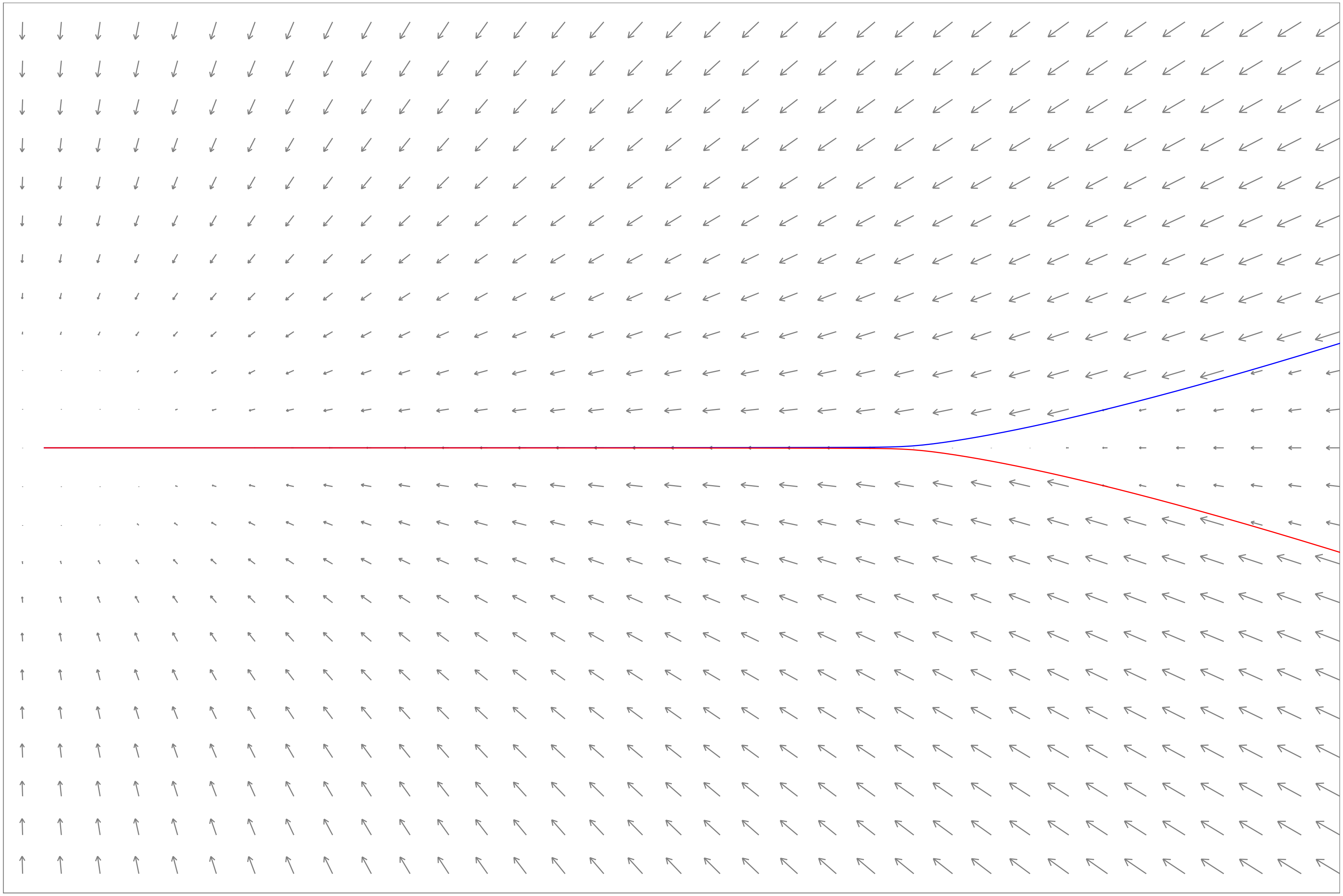
$\mu = 1.3$, $\phi_\infty = 10^\circ$ numerics:



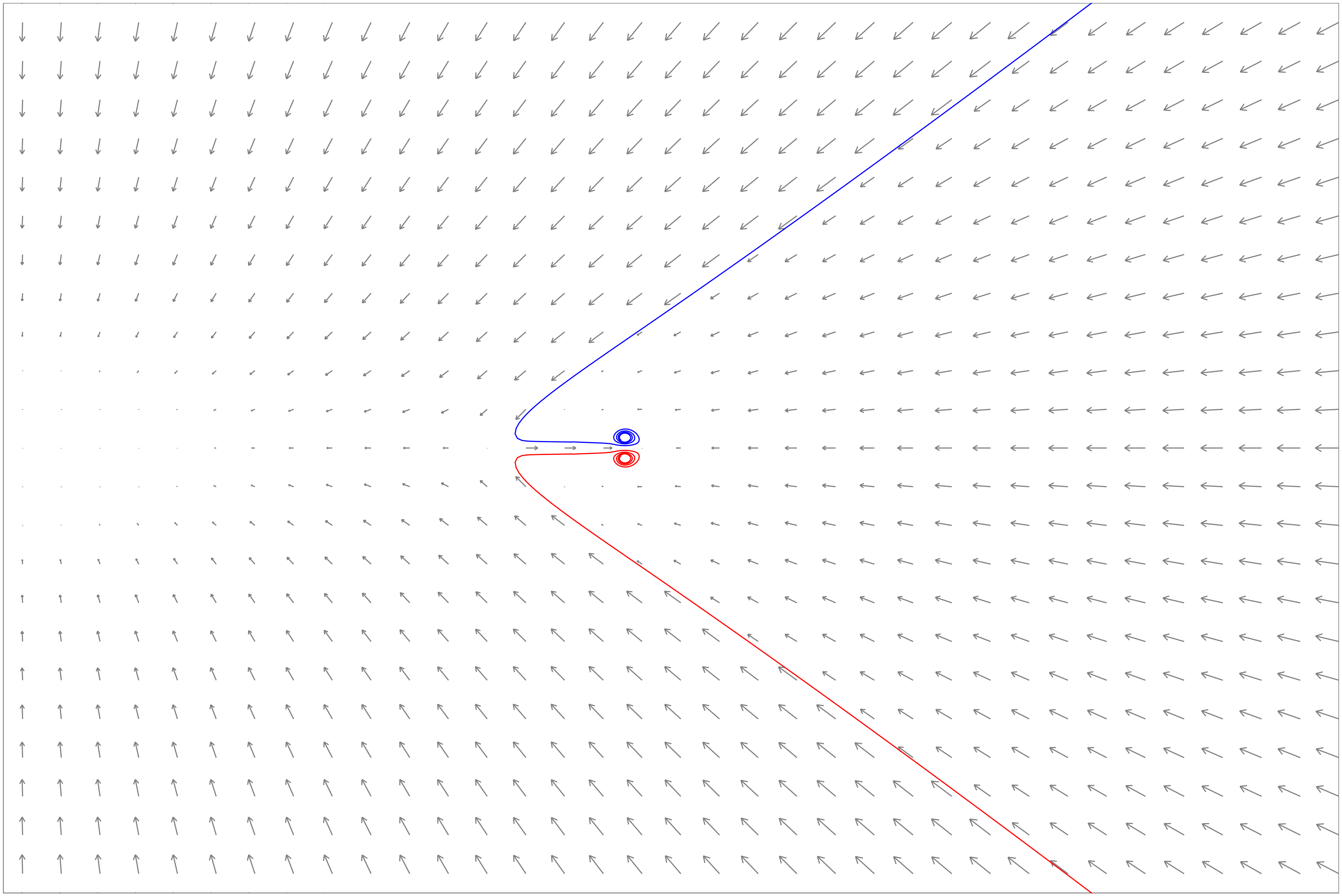
Pseudo-velocity $\mathbf{q}$:



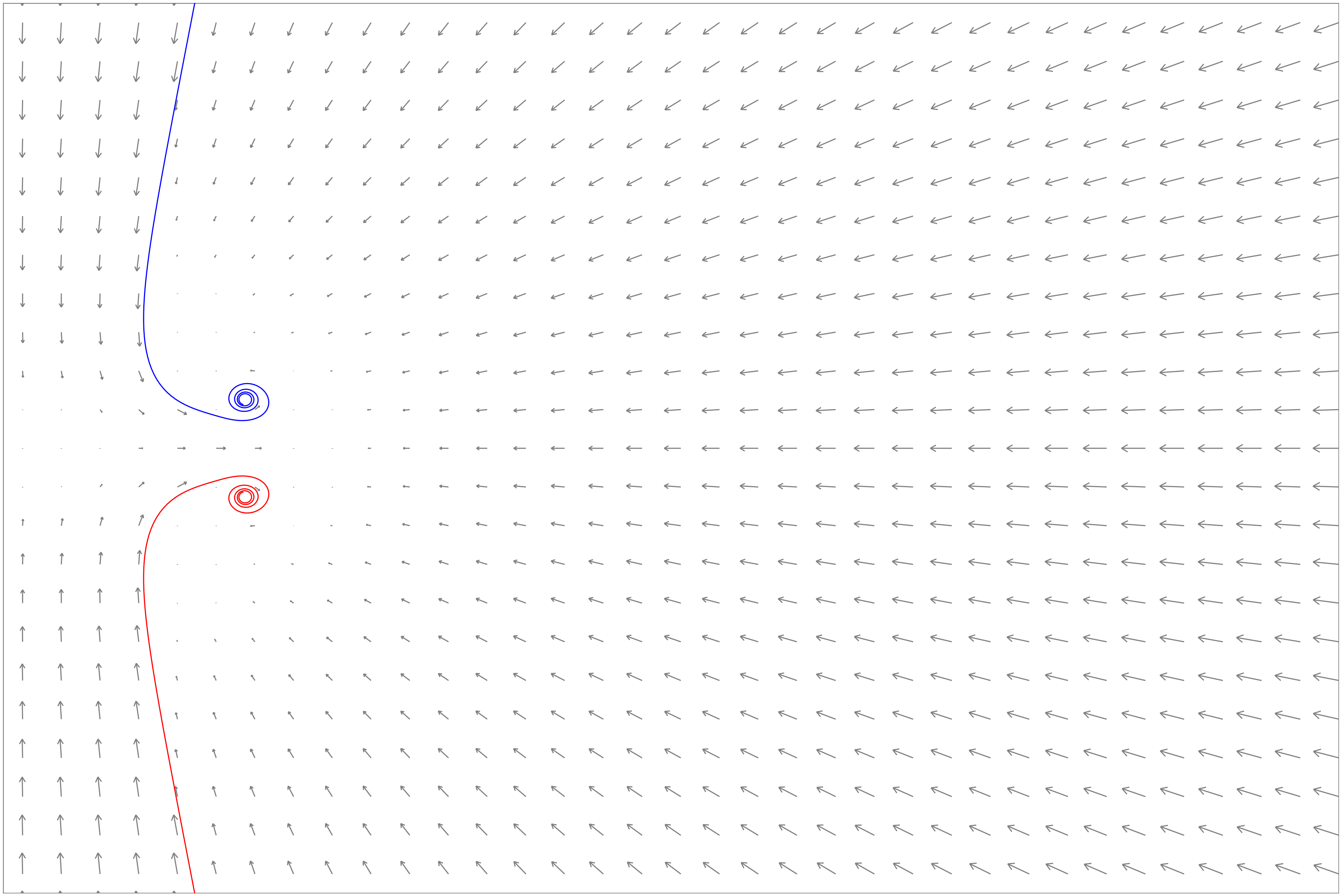
Increase ϕ_∞ to 25°:



Increase ϕ_∞ to 40°:

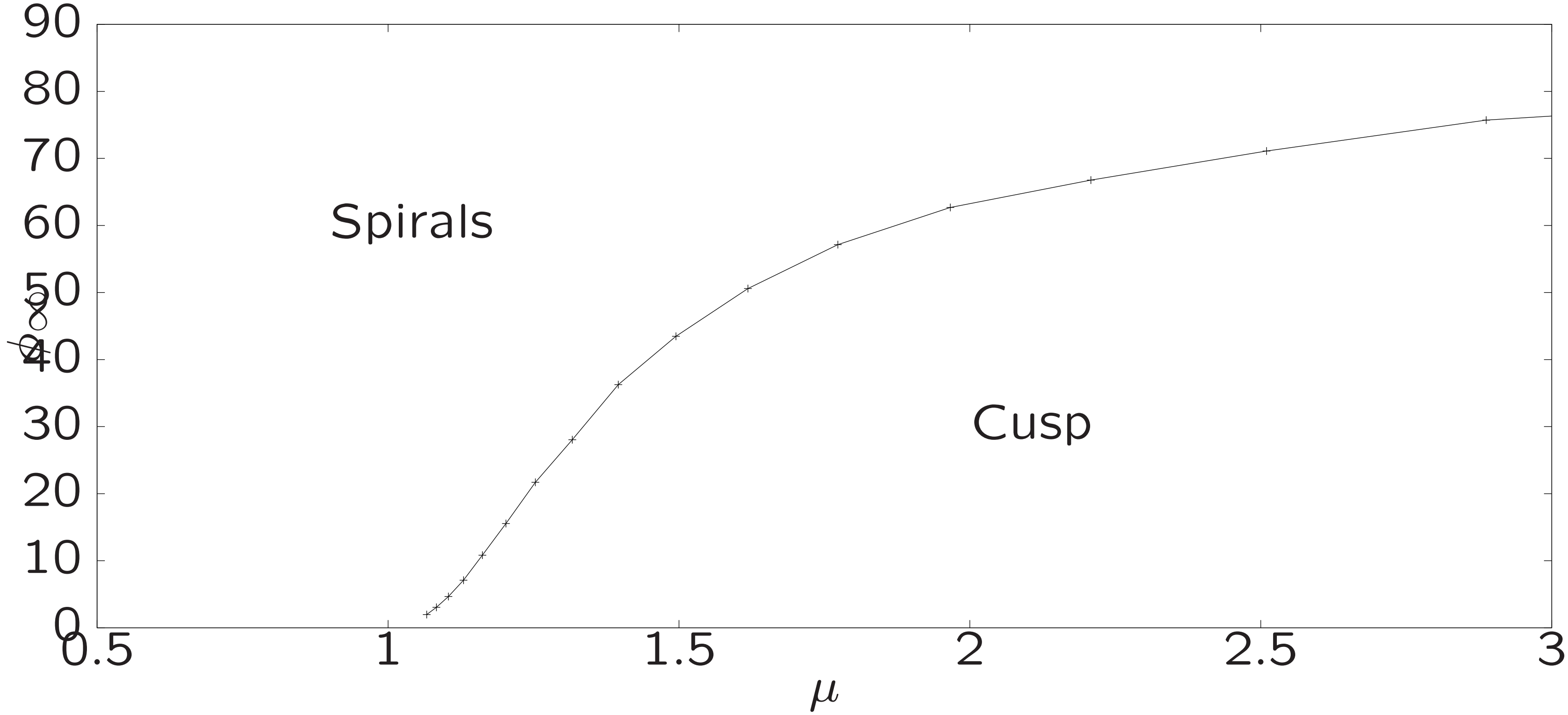


Increase to 80°:



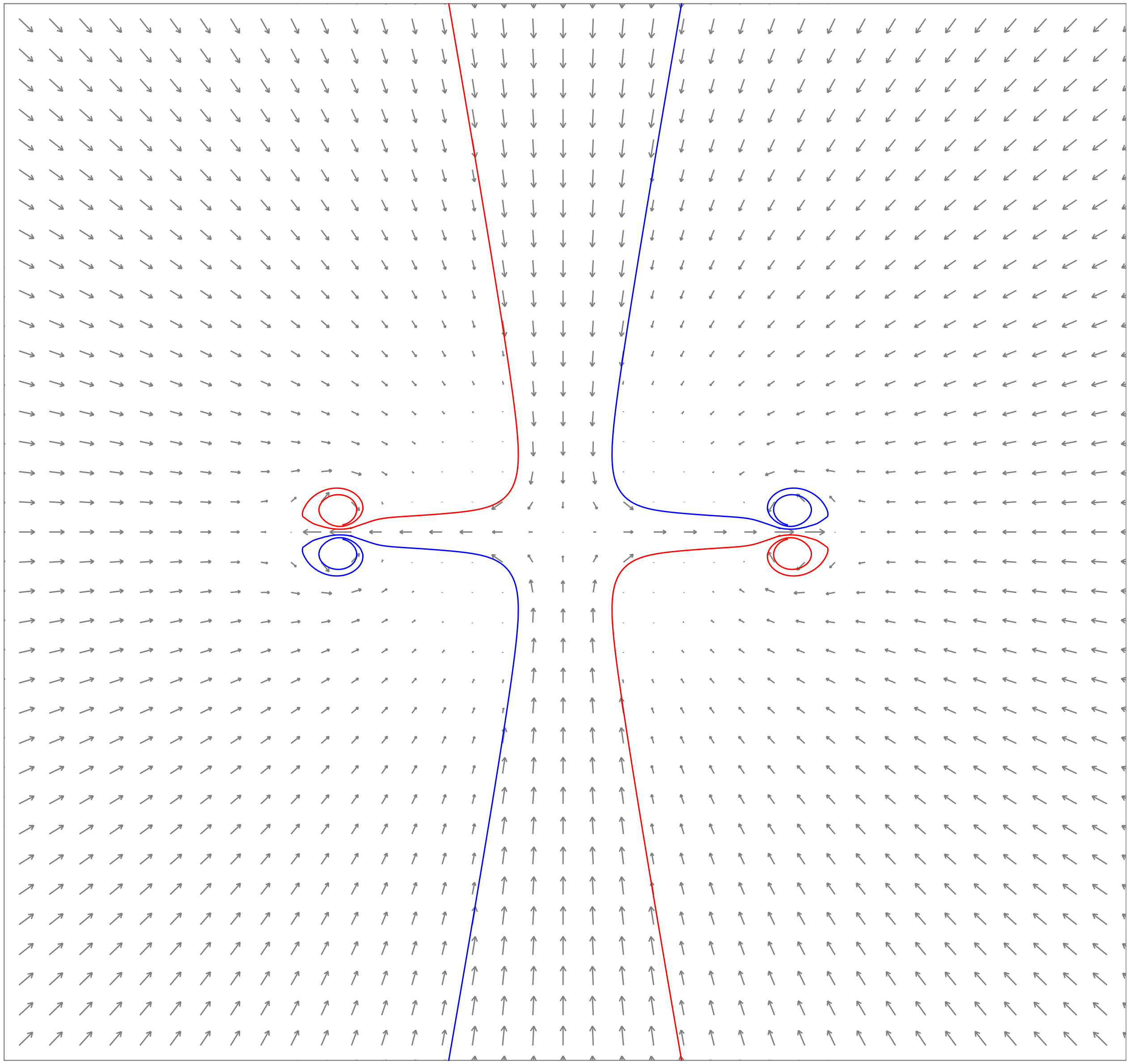
Maximum cusp angle at infinity ϕ_∞

Numerics:

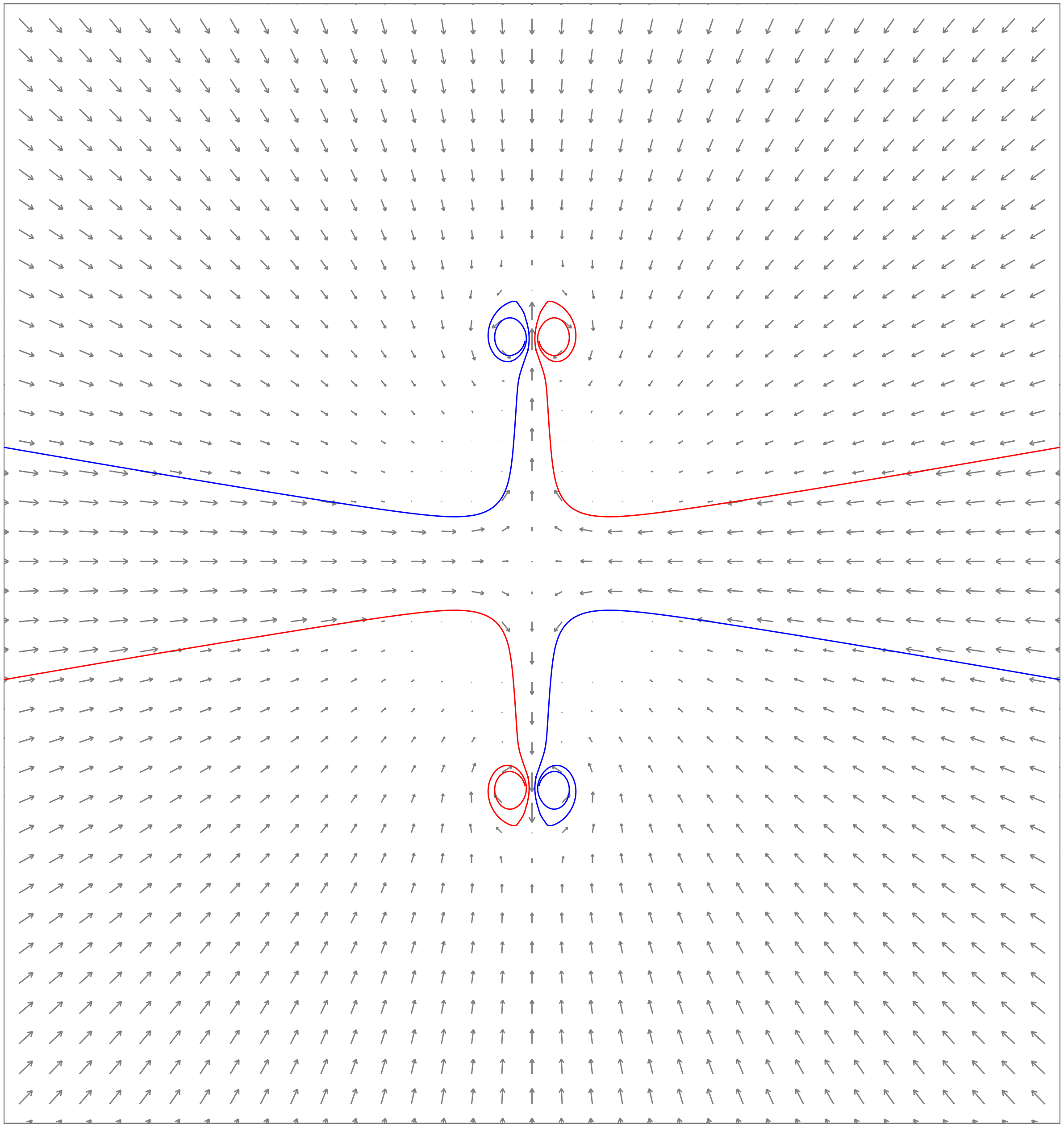


- for $\mu \nearrow \infty$, max angle $\nearrow 90^\circ$
- for $\mu \searrow 1$, max angle $\searrow 0$
- Cusp if $\mu > 1$ and angle below max, otherwise: spirals

$\mu = 1.3$ with $\phi_\infty = 80^\circ$ solution:



Rotate: solution for $\mu = 1.3$ with $\phi_\infty = 10^\circ$, *opposite* circulation



Concluding remarks

- "Most" vortex cusps appear to have asymptotics

$$v^x = x + \ldots \quad , \quad \hat{y} = x^\alpha + \ldots \quad , \quad \boxed{\alpha = \frac{\mu+1}{\mu-1}} \quad \text{as } x \searrow 0$$

- No cusps unless $\mu > 1$ *and* sufficiently small angles-at-infinity *and* positive (ccw) circulation on the upper (second in ccw direction) sheet! (= inner $\mathbf{v}$ points away from cusp.)
  Single Mach reflection does produce such circulation.

- For larger angles, $\mu \le 1$, or negative circulation: numerics suggest non-cusp flows with algebraic spiral ends.

- $\mu = 1$ key (compressible) but *borderline*: may need to superimpose harmonic strain flow to modify exponent.

- Low-order numerics not necessarily stable, yield wrong exponent.