Asymptotic stability for the gradient flow of nonlocal energies

Nicola Fusco

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Tainan, January 23-28, $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4$

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$$P(E) + \text{ Volume term (nonlocal)}$$

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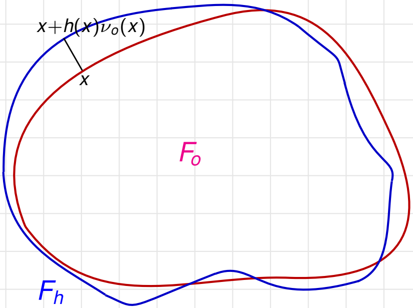
Mullins (1957,1958,1960), Davì-Gurtin (1990)

Evolution of a two phase interface controlled by mass diffusion within the surface

The H^{-1} flow of the perimeter

Cahn-Taylor (1994)



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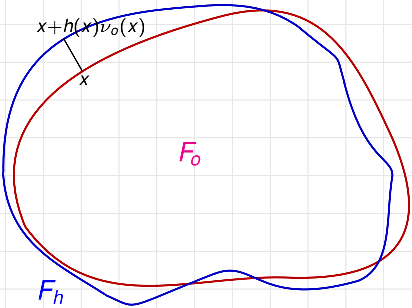

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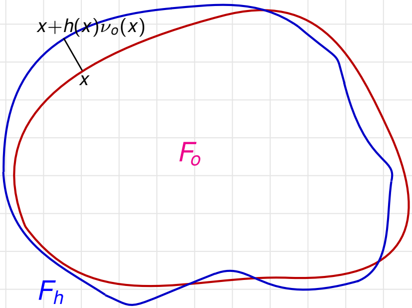

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$$\min\left\{P(F_h) + \frac{1}{2\tau_N}\|h - h_{i,N}\|^2_{H^{-1}(\Gamma_o)} : \|h\|_{C^1(\Gamma_o)} \le M\right\}$$

How is the H^{-1} norm defined?

$$\|h - h_{i,N}\|^2_{H^{-1}} := \int_{\Gamma_{i,N}} |\nabla_{\Gamma_{i,N}} v_h|^2 \, d\mathcal{H}^{n-1}$$

where
$$\begin{cases} \Delta_{\Gamma_{i,N}} v_h = ((h - h_{i,N}) \circ \pi_o)\langle \nu_o, \nu_{\Gamma_{i,N}} \rangle & \text{on } \Gamma_{i,N} \\ \int_{\Gamma_{i,N}} v_h \, d\mathcal{H}^{n-1} = 0 \end{cases}$$

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If the above discrete scheme 'converges' to a function $h(x, t)$

$$\int_{\Gamma_t} H_t \, \varphi \circ \pi_o - \int_{\Gamma_t} w(\cdot, t) \, \varphi \circ \pi_o = 0 \qquad \forall \varphi \in C^1(\Gamma_o)$$

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The same argument with L^2-norm $\Longrightarrow$ $V_t=-H_t$ (mean curvature flow)

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- Surface diffusion is volume preserving

$$\frac{d}{d_t}|F_t| = \int_{\partial F_t} V_t \, d\mathcal{H}^{n-1} = \int_{\partial F_t} \Delta_{\Gamma_t} H_t \, d\mathcal{H}^{n-1} = 0$$

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- Surface diffusion (and mean curvature flow) reduce the perimeter

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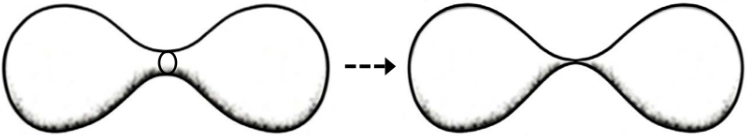
- Surface diffusion does not preserve convexity

  Mean curvature flow preserves convexity and shrinks a convex set to a point in finite time, so that by rescaling the evolving sets to the original volume, they converge to a ball (Huisken, 1984)

Singularities may appear in finite time even in 2-D (Giga-Ito, 1998)



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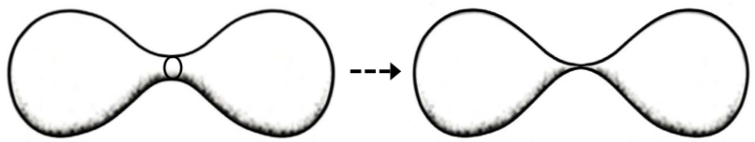

- Existence for small times (Escher-Mayer-Simonett, 1998)

$$F_o \in C^{2,\alpha} \Longrightarrow h \in C^0([0,T); C^{2,\alpha}(\Gamma_o)) \cap C^\infty((0,T); C^\infty(\Gamma_o))$$

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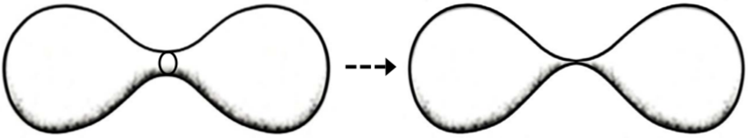

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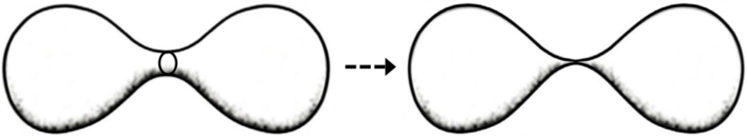

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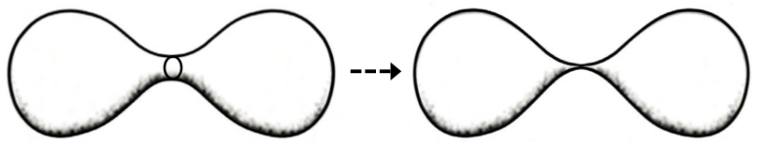

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- $n = 3$

  F_o close to an infinite cylinder (LeCrone, Simonett, 2016)

Evolution of periodic structures (pattern formation)

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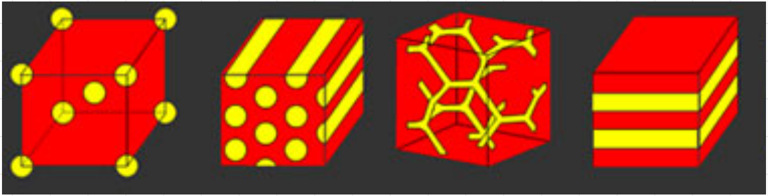

$n = 3$ Periodic sets with constant mean curvature boundary

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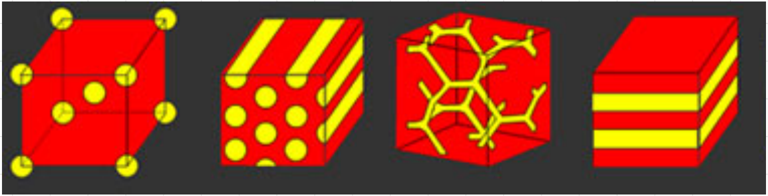

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$$J(F) := P_{\mathbb{T}^n}(F)$$

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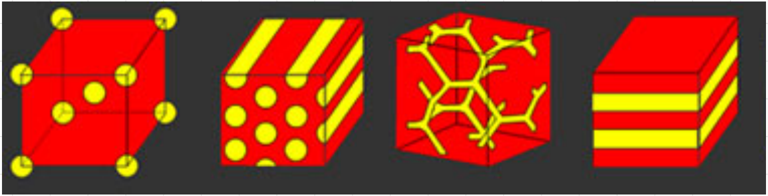

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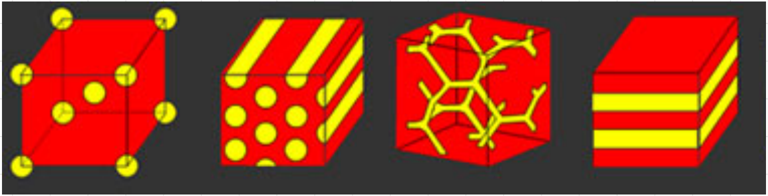

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Given a C^2 vector field $X : \mathbb{T}^n \mapsto \mathbb{T}^n$ let us now define

$$\partial^2 J(F)[X]$$

Consider the flow $\Phi : \mathbb{T}^n \times (-1,1) \mapsto \mathbb{T}^n$

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Thus for a C^2 critical point F and for $\varphi \in H^1(\partial F)$ we set

$$\partial^2 J(F)[\varphi] = \int_{\partial F} \left(|\nabla \varphi|^2 - |B_{\partial F}|^2 \varphi^2 \right) d\mathcal{H}^{n-1}$$

$$\widetilde{H}^1(\partial F) := \left\{ \varphi \in H^1(\partial F) : \underbrace{\int_{\partial F} \varphi = 0}_{\text{volume pres.}}, \quad \underbrace{\int_{\partial F} \varphi \nu_F = 0}_{\text{translation inv.}} \right\}$$

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Theorem (Acerbi-F.-Morini 2013)

Let F be a strictly stable C^2 critical configuration.

Then, F is a strict local minimizer, i.e., there exists $\delta, C_0 > 0$, s.t. if $\min_\tau |F\Delta(\tau + G)| < \delta$

$$J(G) \geq J(F) + C_0 |F\Delta(\tau + G)|^2$$

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The local minimality w.r.t. L^∞ perturbations (B.White, 1994)
or w.r.t. L^1 perturbations ($n \leq 7$, Morgan-Ros, 2010)
In both cases there was no quantitative estimate

Theorem (Acerbi, F., Julin, Morini, JDG to appear)

Let $G \subset \mathbb{T}^3$ be a smooth strictly stable critical set. For every $M > 0$ there exists $\delta > 0$ s.t.:

If $\partial F_o = \{x + h_o(x)\nu_G : \ x \in \partial G, \ \|h_o\|_{H^3(\partial G)} \leq M\}$,

$$|F_o| = |G|\,, \qquad |F_o \Delta G| \leq \delta\,, \qquad \text{and} \qquad \int_{\partial F_o} |\nabla H_{\partial F_o}|^2 \, d\mathcal{H}^2 \leq \delta\,,$$

then the unique classical solution $(F_t)_t$ to the surface diffusion flow with initial datum F_o exists for all $t > 0$.

Moreover, $F_t \to G + \sigma$ in $W^{3,2}$ as $t \to +\infty$, for some $\sigma \in \mathbb{R}^3$.

The convergence is exponentially fast, i.e., there exist η, $c_G > 0$ such that for all $t > 0$, writing

$$\partial F_t = \{x + \psi_{\sigma,t}(x)\nu_{G+\sigma}(x) : \ x \in \partial G + \sigma\}\,,$$

we have

$$\|\psi_{\sigma,t}\|_{H^3(\partial G+\sigma)} \leq \eta e^{-c_G t}\,.$$

Both $|\sigma|$ and η vanish as $\delta \to 0^+$.

Idea of the proof

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\partial F_t} |\nabla_\tau H_t|^2\, dx\right) = -\,\partial^2 J(F_t)\,[\Delta_\tau H_t] - \int_{\partial F_t} B_t\,[\nabla_\tau H_t]\,\Delta_\tau H_t\, d\mathcal{H}^2$$
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But if F_t is sufficiently close to the stable critical point G then

$$\partial^2 J(F_t)\,[\Delta_\tau H_t] \geq c_0 \|\Delta_\tau H_t\|^2_{H^1(F_t)}$$

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But if F_t is sufficiently close to the stable critical point G then

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$$\Downarrow$$

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Idea of the proof

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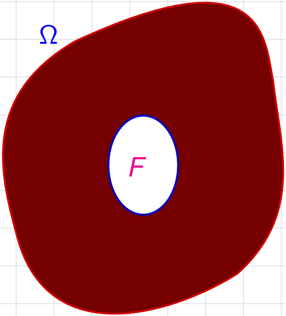
Evolution of material voids

Material void inside a stressed elastic material
(Siegel-Miksis-Voorhees 2004)



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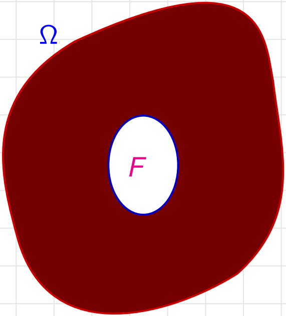

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$\Omega \setminus F =$ the region occupied by the material

$F =$ the void

$u_F : \Omega \setminus F \mapsto \mathbb{R}^3 =$ the elastic equilibrium

$$u_F = \operatorname{argmin} \left\{ \int_{\Omega \setminus F} W(E(u))\, dx : \; u = u_o \text{ on } \partial\Omega \right\}$$

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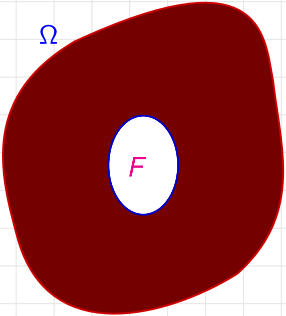

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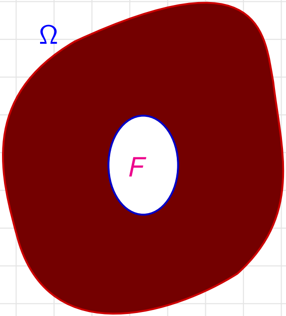

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where $\mathbb{C}$ is a tensor such that $\mathbb{C}A : A > 0$ for all $A \neq 0$

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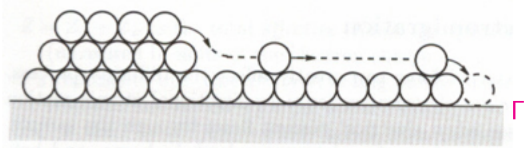
Existence and regularity in 2D (Fonseca-F-Leoni-Millot, 2011)

Morphology evolution: surface diffusion

$$J(F) = \int_{\Omega\setminus F} W(E(u_F))\,dx + \mathcal{H}^2(\partial F)$$



$\Gamma_t = \partial F_t$

Einstein-Nernst law: surface flux of atoms $\propto \nabla_{\Gamma_t}\mu$

μ= chemical potential $\rightsquigarrow$ $V_t = \kappa\,\Delta_{\Gamma_t}\mu$

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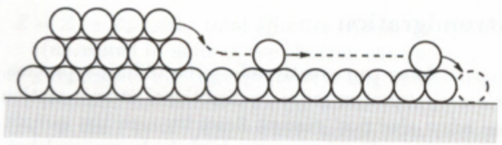

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μ= first variation of energy = $H_t - W(E(u_t)) + \lambda$

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- This is the H^{-1} flow of $J(F)$
- The flow is volume preserving (no information on the perimeter)
- No existence results available!

Theorem (F.-Julin-Morini, 2018)

Let $G \subset\subset \Omega \subset\subset \mathbb{R}^3$ *smooth. For every* $M > 0$ *there exist* $\delta > 0,\ T > 0$ *s.t. if*

$$\partial F_o = \left\{x + h_o(x)\nu_{_G} :\ x \in \partial G,\ \|h_o\|_{_{H^3(\partial G)}} \le M\right\}, \qquad \|h_o\|_{L^2(\partial G)} \le \delta,$$

then there exists a unique solution $(F_t)_t$, $t \in (0, T)$. *More precisely*

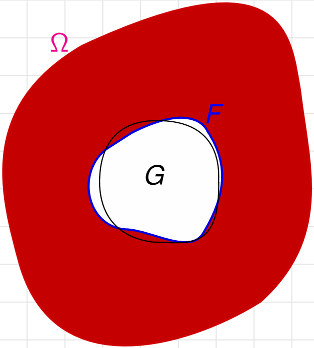
$$\partial F_t = \{x + h(x,t)\nu_{_G}(x) :\ x \in \partial G\}$$

where

$$h \in L^\infty((0,T); H^3(\partial G)) \cap H^1((0,T); H^1(\partial G))$$

Moreover, for all integers $k \ge 0$,

$$\sup_{0 \le t \le T} t^k \|h(\cdot,t)\|^2_{H^{2k+3}(\partial G)} + \int_0^T t^k \|h(\cdot,t)\|^2_{H^{2k+5}(\partial G)}\, dt \le C(k, M).$$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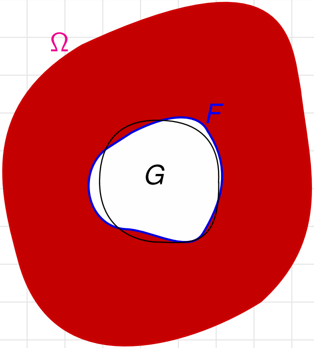

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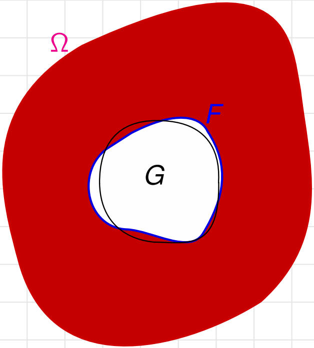

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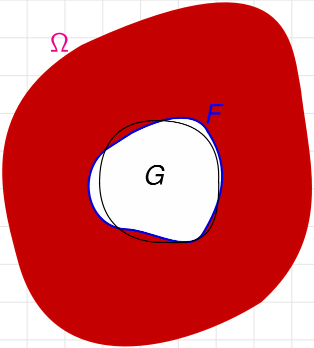

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Theorem

Let $K > 0$, $\alpha \in (0,1)$, and let $k \geq 3$ be an integer. There exists $C_k = C_k(K) > 0$ such that if $h \in H^k(\partial G)$, $\|h\|_{C^{1,\alpha}} \leq K$ then

$$\|W(E(u_{F_h})) \circ \lambda_h\|_{H^{k-\frac{3}{2}}(\partial G)} \leq C_k(\|h\|_{H^k(\partial G)} + 1)$$

Moreover there exists $C = C(K) > 0$ such that, if h_1, $h_2 \in H^3(\partial G)$ with $\|h_i\|_{H^3(\partial G)} \leq K$, for $i = 1, 2$, then

$$\|u_{F_{h_2}} \circ \lambda_{h_2} - u_{F_{h_1}} \circ \lambda_{h_1}\|_{H^{3/2}(\partial G)} \leq C\|h_2 - h_1\|_{H^2(\partial G)}$$

Strictly stable critical points

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Fix $X \in C^2_c(\Omega; \mathbb{R}^3)$, with $\operatorname{div} X = 0$ in a nhood of ∂F

Consider the flow $\Phi : \Omega \times (-\varepsilon, \varepsilon) \mapsto \Omega$

$$\frac{\partial \Phi}{\partial t} = X(\Phi), \qquad \Phi(x, 0) = x$$

and set $F_t := \Phi(\cdot, t)(F)$

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$$\partial^2 J(F)[X] := \frac{d^2}{dt^2} J(F_t)\Big|_{t=0}$$

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$$\partial^2 J(F)[X] = \int_{\partial F} |\nabla(X\cdot\nu)|^2 - |B_F|^2 (X\cdot\nu)^2 \, d\mathcal{H}^2 - 2\int_{\Omega\setminus F} W(E(w_X))\,dx$$

$$- \int_{\partial F} \partial_\nu (W(E(u_F)))(X\cdot\nu)^2 \, d\mathcal{H}^2 - \int_{\partial F} (H_{\partial F} - W(E(u_F)))\,\mathrm{div}_{\partial F}((X\cdot\nu)X_\tau)\, d\mathcal{H}^2$$

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F is strictly stable if for all $X \neq 0$ with $\mathrm{div}\,X = 0$ in a nhood of ∂F

$$\partial^2 J(F)[X] > 0$$

Long time existence

Theorem (F-Julin-Morini, 2018)

Let $G \subset\subset \Omega$ be a smooth strictly stable critical point.

There exists $\delta > 0$ such that if $F_o \subset \Omega$ satisfies

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Moreover $F_t \to G$ H^3-exponentially fast.

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But we can say more. . . . . .

Denote by $\Gamma_1, \ldots, \Gamma_m$ the connected components of ∂G

and by $\mathcal{O}_1, \ldots \mathcal{O}_m$ the open sets enclosed by the Γ_i

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$$\text{then} \qquad F_t \to F_\infty \qquad \text{in } H^3$$

where F_∞ is the only stationary point H^3-close to G s.t.

$$|\mathcal{O}_{i,\infty}| = |\mathcal{O}_{i,o}| \qquad \forall i = 1, \ldots, m$$

THANK YOU FOR YOUR ATTENTION!