Hyperbolic Balance Laws for Multilane Traffic Flow Model

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Content

- History: LWR mode $\Rightarrow$ PW model $\Rightarrow$ AR model
- Goals: Multilane model (applying AR model)

- **LWR model** Lightihll, Whitham (1955) & Richards (1956)

  $\rho_t + (\rho v_e(\rho))_x = 0$, where $q_e(\rho) = \rho v_e(\rho)$.

  (conservation law of vehicles)

  We have two important results if traffic is in equilibrium.

  1. (Anisotropy) $v_e'(\rho) \leq 0 \Rightarrow q_e'(\rho) = v_e(\rho) + \rho v_e'(\rho) \leq v_e(\rho)$.
  2. (Acceleration) $a = dv/dt = -\rho_x (v_e'(\rho))^2 \rho$

equilibbrium		
ρ: *small* $\Rightarrow$ v: *large*	•	ρ: *large* $\Rightarrow$ v: *small*
	$\rho_x > 0 \Rightarrow a < 0$	

Drawbacks of LWR model: Traffic is not in equilibrium

1. Instability at the vacuum : slow drivers in the light traffic
2. drivers' behavior : $a = v_x(\ldots)$, not $a = -\rho_x(\ldots)$

non − equilibbrium		
ρ: *small*, v: *small*	•	ρ: *large*, v: *large*
$v_x > 0 \Rightarrow a > 0$		

cf.

equilibbrium		
ρ: *small* $\Rightarrow$ v: *large*	•	ρ: *large* $\Rightarrow$ v: *small*
$\rho_x > 0 \Rightarrow a < 0$		

- **PW moel** Payne (1971) & Whitham (1974)

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + v v_x + \frac{A_e'(\rho)}{\rho} \rho_x = \frac{v_e(\rho) - v}{\tau}. \end{cases}$$

NOTE. Different choices on $A_e(\rho)$, "pressure" of traffic

1. Payne $A_e'(\rho) = \frac{1}{2\tau} |v_e'(\rho)|$
2. Whitham $A_e'(\rho) = D$
3. Zhang (1998): $A_e'(\rho) = (\rho v_e'(\rho))^2$
4. Others ...

- **But ...**

$$\text{PW model} \begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + v v_x + \frac{A_e'(\rho)}{\rho} \rho_x = \frac{v_e(\rho) - v}{\tau}. \end{cases}$$

Drawbacks of PW model (Daganzo, 1995)

- Violate anisotropy : $v + \sqrt{A_e'(\rho)}$ (wave) $> v$ (vehicle)
- Diffusion: negative speed

Ref. Daganzo, **Requiem** (安魂曲) for second-order fluid approximation of traffic flow, Transportation Res., Part B, 1995.

Fortunately, AR model overcomes these drawbacks Daganzo mentioned.

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ \alpha_t + v\alpha_x \quad = 0, \end{cases}$$

where $\alpha = v + c\rho^\gamma$ ($\gamma > 0$).

This model obeys

- anisotopy,
- instability at the vacuum,
- non-diffusion effect
- reasonable drivers' behavior

Ref. Aw and Rascle, **Resurrection** (復活，Risurrezione) of "second" order models of traffic flow, SIAM, J. Appl. Math, 2000.

A Multilane Model (Greenberg, Klar and Rascle, 2003)

- For an unidirectional one-dimensional road with n lanes, the macroscopic variables are the density ρ of vehicles and the average speed v across all the lanes. Then

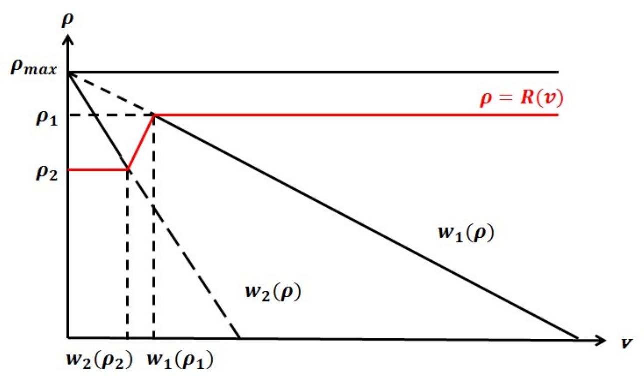
$$\rho = \Sigma_{i=1}^{i=n} \rho_i \quad \text{and} \quad \rho v = \Sigma_{i=1}^{i=n} \rho_i v_i \text{ (total flux)},$$

  where ρ_i and v_i are respectively the density and the average speed of vehicles in the i-th lane.

- When traffic is high, lane changing and passing is difficult.
  $\Rightarrow$ the equilibrium speed for vehicles is low.

- When traffic is lower, these actions become easier.
  $\Rightarrow$ the equilibrium speed for vehicles is higher.

- With the aid of Kerner's three-phase traffic theory, Greenberg, Klar, and Rascle give the following figure.



- For simplicity, we apply Sopasakis' argument (2002): $\rho_1 = \rho_2$.

- Multilane model (Greenberg, Klar, and Rascle, 2003)

$$\begin{cases} \rho_t + (m - c\rho^2)_x = 0 \\ m_t + (\frac{m}{\rho}(m - c\rho^2))_x = \begin{cases} \frac{\rho w_1(\rho) - (m - c\rho^2)}{\tau}, & \rho < \rho_*, \\ \frac{\rho w_2(\rho) - (m - c\rho^2)}{\tau}, & \rho \geq \rho_*, \end{cases} \end{cases}$$

where $m \equiv \rho v + c\rho^2$ and $\rho \neq 0$.

- Multilane model (balance laws) $U_t + F(U)_x = G(U)$

$=$ AR model (conservation laws) $U_t + F(U)_x = 0$

$+$ Kerner's theory, 1998 (source term) $G(U)$

- Our **goals** are:

(1) to solve Riemann problem for the AR model

(2) to get an approximate solution for the multilane model.

Definition (Riemann's problem)

The Riemann problem is the initial-value problem for the conservation laws

$$U_t + F(U)_x = 0 \text{ in } \mathbb{R} \times (0, \infty)$$

with the piecewise-constant initial data

$$U(x,0) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0. \end{cases}$$

We call u_L and u_R the left and right initial data, respectively.

U_L (*upstream*) $\longrightarrow$ U_R (*downstream*)

- Since the flux F is a smooth function, the system we consider is of the form $U_t + DF(U)U_x = 0$. More precisely,

$$\begin{bmatrix} \rho \\ m \end{bmatrix}_t + \begin{bmatrix} -2c\rho^2 & 1 \\ -(\frac{m}{\rho})^2 - mc & \frac{2m}{\rho} - c\rho \end{bmatrix} \begin{bmatrix} \rho \\ m \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

this gives the distinct real eigenvalues of $DF(U)$

$$\lambda_1(U) = \frac{m}{\rho} - 2c\rho < \lambda_2(U) = \frac{m}{\rho} - c\rho \ (= v)$$

so that the system is strictly hyperbolic since $\rho \neq 0$.

Thus this model is anisotropic.

- Moreover, the corresponding eigenvectors can be taken as

$$r_1(U) = \frac{-1}{2c}\begin{bmatrix} 1 \\ \frac{m}{\rho} \end{bmatrix} \text{ and } r_2(U) = \frac{-1}{2c}\begin{bmatrix} 1 \\ \frac{m}{\rho} + c\rho \end{bmatrix}.$$

Definition

The kth characteristic field is said to be genuinely nonlinear if

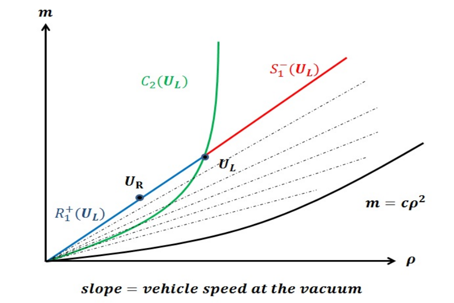
$$\nabla\lambda_k(z) \cdot r_k(z) \neq 0 \text{ for all } z.$$

The kth characteristic field is said to be linearly degenerate if

$$\nabla\lambda_k(z) \cdot r_k(z) = 0 \text{ for all } z.$$

It follows from our system that

- The first characteristic field is genuinely nonlinear since $\nabla\lambda_1(U) \cdot r_1(U) = 1 \neq 0$.
- The second characteristic field is linearly degenerate since $\nabla\lambda_2(U) \cdot r_2(U) = 0$.



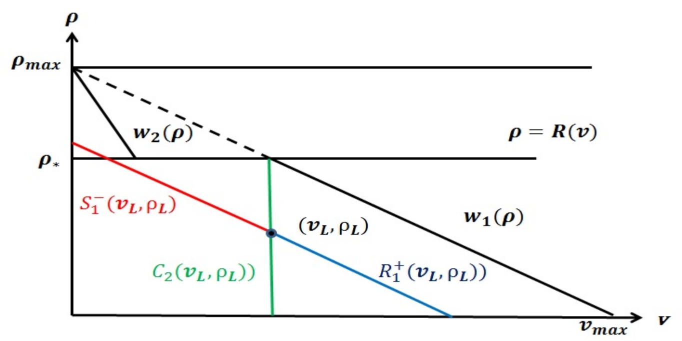
- After some computations we have the three essential curves

  $$R_1^+(U_L),\ S_1^-(U_L) \text{ and } C(U_L).$$

  to help us to solve the Riemann problem of conservation laws.

  $$R_1^+(U_L) = \{(\rho, m)^T : \frac{m_L}{\rho_L} = \frac{m}{\rho}, \rho < \rho_L, m < m_L\},$$

  The slope $= m_L/\rho_L = v + c\rho = v_0 + c \cdot 0$, where v_0 is the vehicle speed at the vacuum.



- The three curves can be expressed in the $v-\rho$ plane. That is,

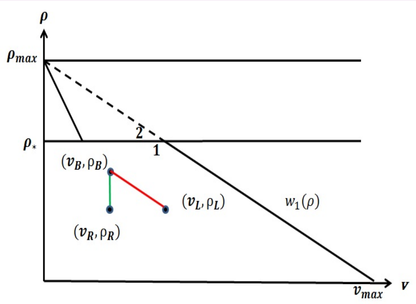
$$R_1^+(v,\rho) = \{(v,\rho)^T : v = v_L + c(\rho_L - \rho), 0 < \rho < \rho_L\},$$
$$S_1^-(v,\rho) = \{(v,\rho)^T : v = v_L + c(\rho_L - \rho), \rho_L \leq \rho \leq \rho_{\max}\},$$
$$C(v,\rho) \ = \{(v,\rho)^T : v = v_L, 0 < \rho \leq \rho_{max}\}.$$

Let $\mathbb{D} = \{(v, \rho) : 0 \leq v \leq w_1(\rho)), 0 < \rho \leq \rho_{\max}, 0 \leq v \leq v_{\max}\}$.
Then it is the triangle region.

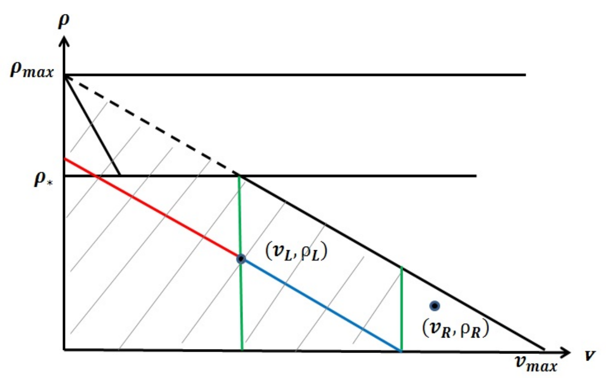
Theorem (Aw and Rascle 2000)

Given the left initial state U_L. If the right initial state U_R is in the region $\mathbb{D}$, then there exists a unique integral solution U of Riemann's problem, which is constant on lines through the origin.

NOTE. This $\mathbb{D}$, the triangle region, is called the invariant region.



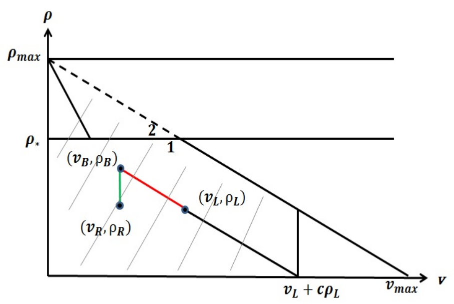
But the Riemann's problem obviously has no solution for such initial data in the following figure.



Fortunately, extending their work, we can define the trapezoid

$$\mathbb{D}(v_L, \rho_L) = \mathbb{D} \cap \{v : 0 \leq v \leq v_L + c\rho_L\}$$

as a new invariant region such that the Riemann's problem has a unique solution for any $(v_R, \rho_R) \in \mathbb{D}(v_L, \rho_L)$.



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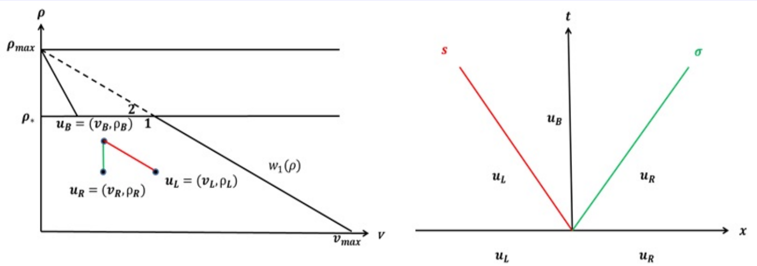


In this case, $u_L^1 \xrightarrow{S} u_B^1 \xrightarrow{C} u_R^1$ denoted by $(1,1,1)_S$.

Example

For $(1,1,1)_S$ we have

$$u(x,t) = \begin{cases} u_L, & x < st, & s = \frac{\lambda_1(u_L)+\lambda_1(u_B)}{2}, \\ u_B, & st < x < \sigma t, & \\ u_R, & \sigma t < x, & \sigma = \frac{\lambda_2(u_B)+\lambda_2(u_R)}{2}. \end{cases}$$

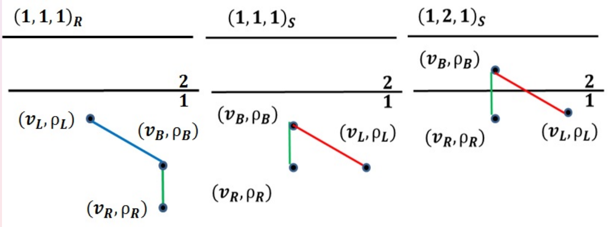
- $u_B = \left(\frac{1}{c}((c\rho_L + v_L)(c\rho_L + v_L - v_R)), \; \frac{1}{c}(c\rho_L + v_L - v_R) \right)$

In general, the horizontal line $\rho = \rho_*$ divides the invariant region $\mathbb{D}(u_L)$ into two subregions, called Ω_1 if $\rho < \rho_*$, and Ω_2 if $\rho_* \leq \rho$. Thus there are four cases that need to be considered in terms of $(u_L^i, u_R^j) \in (\Omega_i, \Omega_j)$, where $i, j = 1, 2$.

(1) For the case $(u_L^1, u_R^1) \in (\Omega_1, \Omega_1)$, there are three possibilities:

$$u_L^1 \xrightarrow{R} u_B^1 \xrightarrow{C} u_R^1, \quad u_L^1 \xrightarrow{S} u_B^1 \xrightarrow{C} u_R^1, \quad \text{and} \quad u_L^1 \xrightarrow{S} u_B^2 \xrightarrow{C} u_R^1,$$

which are denoted by $(1,1,1)_R$, $(1,1,1)_S$, and $(1,2,1)_S$, resp..



(1) $(u_L^1, u_R^1) \in (\Omega_1, \Omega_1)$: $(1,1,1)_R$, $(1,1,1)_S$, $(1,2,1)_S$

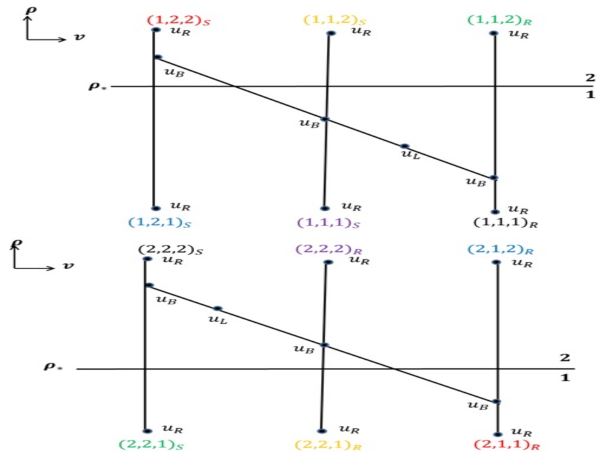
Analogous arguments can be applied to the others. That is,

(2) $(u_L^1, u_R^2) \in (\Omega_1, \Omega_2)$: $(1,1,2)_R$, $(1,1,2)_S$, $(1,2,2)_S$

(3) $(u_L^2, u_R^1) \in (\Omega_2, \Omega_1)$: $(2,2,1)_R$, $(2,1,1)_R$, $(2,2,1)_S$

(4) $(u_L^2, u_R^2) \in (\Omega_2, \Omega_2)$: $(2,2,2)_R$, $(2,1,2)_R$, $(2,2,2)_S$

In fact, there are twelve cases as follows. ($R \longleftrightarrow S$ and $1 \longleftrightarrow 2$)



NOTE. We can get the geometrical information from the algebraic notations $(i, j, k)_{R,S}$, and the converse is also true.

Back to the balance laws (multiland model),

$$\begin{cases} \rho_t + (m - c\rho^2)_x = 0 \\ m_t + (\frac{m}{\rho}(m - c\rho^2))_x = \begin{cases} \frac{\rho w_1(\rho) - (m - c\rho^2)}{\tau}, & \rho < R(\frac{m - c\rho^2}{\rho}), \\ \frac{\rho w_2(\rho) - (m - c\rho^2)}{\tau}, & \rho \geq R(\frac{m - c\rho^2}{\rho}), \end{cases} \end{cases}$$

we consider

$$\begin{cases} \widetilde{U}_t + F(\widetilde{U})_x = 0, \\ \widetilde{U}(x, 0) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0, \end{cases} \end{cases} \quad \& \quad \begin{cases} U_t + F(U)_x = G(U), \\ U(x, 0) = \begin{cases} U_L, & x < 0, \\ U_R, & x > 0. \end{cases} \end{cases}$$

Let $\overline{U} = U - \widetilde{U}$. Then $\overline{U}(x, 0) = 0$ for any nonzero x.

Our goal is to obtain an approximate solution of $\overline{U}$ and use it to get an approximate solution of U.

It follows from $U_t + F(U)_x = G(U)$ and $U = \widetilde{U} + \overline{U}$ that

$$(\widetilde{U} + \overline{U})_t + F(\widetilde{U} + \overline{U})_x = G(\widetilde{U} + \overline{U}).$$

Doing linearization on F and G, and applying the operator-splitting method, the perturbation $\overline{U}$ is the solution of the initial-value problem of ODE.

$$\begin{cases} \overline{U}_t = G(\widetilde{U}) + DG(\widetilde{U})\overline{U}, \\ \overline{U}(x,0) = 0. \end{cases}$$

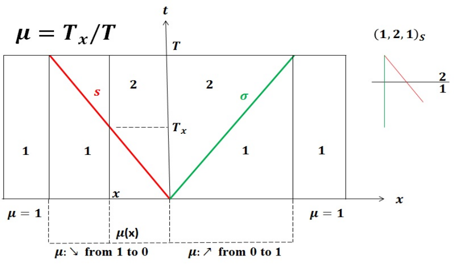
To solve it, we need to introduce a new parameter $\mu(x)$ which depends only on the location x.

Definition

Fixed a location x, the parameter is defined to be the ratio

$$\mu(x) = T_x / T,$$

where T_x is the observed time when the solution $\widetilde{U}$ exists in Ω_1, and T is the observed time when the solution $\widetilde{U}$ exists in $\Omega_1 \cup \Omega_2$.



NOTE. Obviously, the parameter is decreasing from 1 to 0, and increasing from 0 to 1.

Theorem (Approximate Solution)

The approximate solution of the balance laws (multilane model) is

$$\begin{cases} \rho = \widetilde{\rho}, \\ v = \theta \widetilde{v} + (1-\theta) w(\mu, \widetilde{\rho}), \end{cases}$$

where $\theta = e^{-t/\tau}$ *and* $w(\mu, \widetilde{\rho}) \equiv \mu w_1(\widetilde{\rho}) + (1-\mu) w_2(\widetilde{\rho})$.

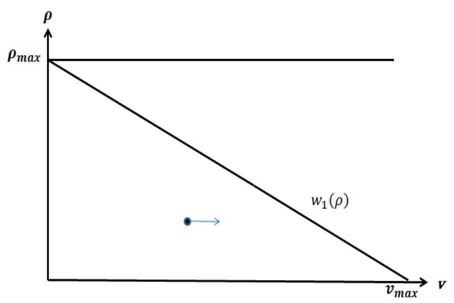
If $\tau \to 0$, then $\theta = e^{-t/\tau} \to 0$, and thus the above theorem shows that the approximate speed v approaches $w(\mu, \widetilde{\rho})$, i.e.,

$$\lim_{\tau \to 0} v = \lim_{\theta \to 0} (1-\theta) w(\mu, \widetilde{\rho}) + \theta \widetilde{v} = w(\mu, \widetilde{\rho}).$$

Clearly, this gives the asymptotic behavior as the relaxation time τ tends to zero.

In particular, if we consider the AR model with a single source term $\frac{w_1 - v}{\tau}$, then our result on the asymptotic behavior is similar to the result of Liu.

- $w_1(\widetilde{\rho}) \geq \widetilde{v} \Rightarrow v \geq \widetilde{v}$
- $\lim_{\tau \to 0} v = w_1(\widetilde{\rho})$, similar to the result of Liu. (1991)



For Solving the initial-value problem of the balance laws

$$\begin{cases} U_t + F(U)_x = G(U), \\ U(x,0) = U_0(x). \end{cases},$$

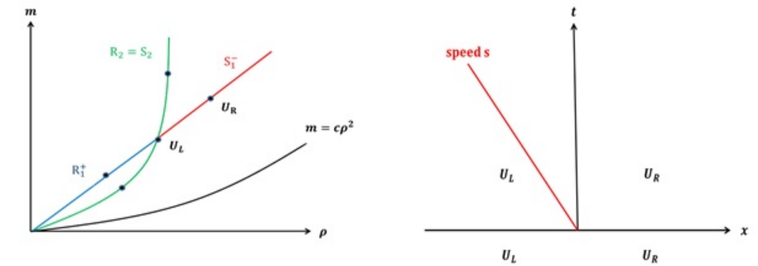
we has three steps.

- Step 1. Solving the Riemann problem for the conservation laws $\widetilde{U}_t + F(\widetilde{U})_x = 0$. $(U = \widetilde{U} + \overline{U})$
- Step 2. Using the solution $\widetilde{U}$ of Riemann problem to obtain an approximate solution for the solution U of balance laws.
- Step 3. Apply the approximate solutions to construct the building blocks for the generalized Glimm scheme to solve the initial-value problem for the balance laws.

Thank you

Thank you

Thank you



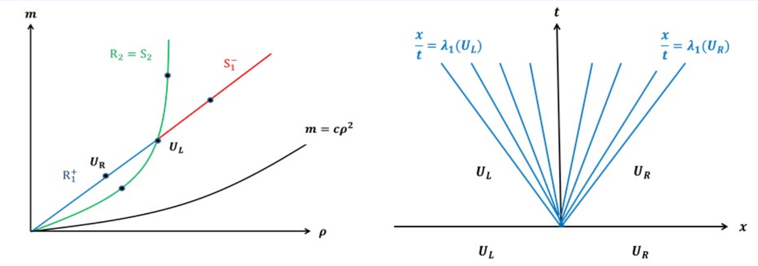
Example (Shock Wave)

Given U_L, if U_R lies in S_1^-, then

$$U(x,t) = \begin{cases} U_L, & x < st, \\ U_R, & x > st, \end{cases}$$

where $s = \frac{m_L}{\rho_L} - c(\rho_R + \rho_L)$.

NOTE. Shock speed $s = \frac{\lambda_1(U_L)+\lambda_1(U_R)}{2}$.

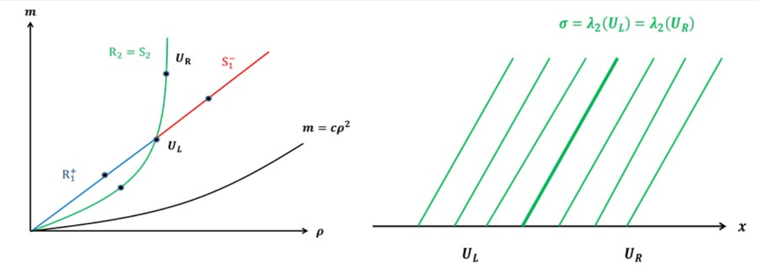


Example (Rarefaction Wave)

Given U_L, if U_R lies in R_1^+, then

$$U(x,t) = \begin{cases} U_L, & \frac{x}{t} \leq \lambda_1(U_L), \\ V(\frac{x}{t}), & \lambda_1(U_L) \leq \frac{x}{t} \leq \lambda_2(U_R), \\ U_R, & \frac{x}{t} \geq \lambda_2(U_R). \end{cases}$$

NOTE. $V(\frac{x}{t}) = \left[\begin{array}{cc} \frac{1}{2c}\left(\frac{m_L}{\rho_L} - \frac{x}{t}\right) & \frac{1}{2c}\frac{m_L}{\rho_L}\left(\frac{m_L}{\rho_L} - \frac{x}{t}\right) \end{array} \right]^T$



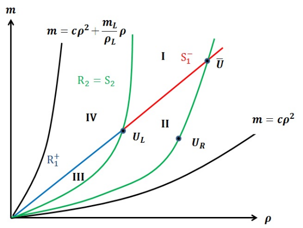
Example (2-contact discontinuity)

Given U_L, if U_R lies in $S_2(= R_2)$, then

$$U(x,t) = \begin{cases} U_L, & x < \sigma t, \\ U_R, & x > \sigma t, \end{cases}$$

where $\sigma = \lambda_2(U_L) = \lambda_2(U_R)$.

NOTE. $\lambda_2(U) = \frac{m}{\rho} - c\rho = v$, the average speed of vehicles.



- Define $\epsilon_1 = -2((\frac{m_L}{\rho_L} - c\rho_L) - (\frac{m_R}{\rho_R} - \rho_R))$ and $\epsilon_2 = \frac{m_L}{\rho_L} - \frac{m_R}{\rho_R}$. Then the next result $(U_L \to \overline{U} \to U_R)$ is an application of the previous theorem.

	I	II	III	IV
ε_1	−	−	+	+
ε_2	−	+	+	−

- NOTE. $\begin{cases} \overline{U} = U_L + \epsilon_1 r_1(U_L), \\ U_R = \overline{U} + \epsilon_2 r_1(\overline{U}) + \frac{\epsilon_2^2}{2} Dr_k(\overline{U}) \cdot r_k(\overline{U}). \end{cases}$

- For Riemann's problem, we have in general

Theorem (local solution of Riemann problem)

Assume each kth characteristic field is either genuinely nonlinear or linearly degenerate. Suppose further that the left initial state U_L is given. Then for each right initial state U_R sufficiently close to U_L there exists an integral solution U of Riemann's problem, which is constant on lines through the origin.

- To our system,

$$\begin{bmatrix} \rho \\ m \end{bmatrix}_t + \begin{bmatrix} -2c\rho^2 & 1 \\ -(\frac{m}{\rho})^2 - mc & \frac{2m}{\rho} - c\rho \end{bmatrix} \begin{bmatrix} \rho \\ m \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we can give a complete solution of Riemann's problem for any two "reasonable" constant states ($U_L = (\rho_L, m_L)^T$ and $U_R = (\rho_R, m_R)^T$).

- To our system, we have the following theorem.

Theorem

Assume that each kth characteristic field is normalized, that is, $\nabla\lambda_k(U)\cdot r_k(U) = 1$ *and* $l_k(U)\cdot r_k(U) = 1$ *for each* k*, and the left initial state* U_L *is given. Then*

(**a**) *there exists a parametrization of* $R_1^+(U_L)$:
$\epsilon \mapsto U_L + \epsilon r_1(U_L)$, *where* $\epsilon = 2c(\rho_L - \rho) > 0$,

(**b**) *there exists a parametrization of* $S_1^-(U_L)$:
$\epsilon \mapsto U_L + \epsilon r_1(U_L)$, *where* $\epsilon = 2c(\rho_L - \rho) < 0$,

(**c**) *there exists a parametrization of* $R_2(U_L) = S_2(U_L)$:
$\epsilon \mapsto U_L + \epsilon r_2(U_L) + \frac{\epsilon^2}{2} Dr_2(U_L)\cdot r_2(U_L)$, *where* $\epsilon = c(\rho_L - \rho)$.

- As a consequence of the previous result, we give a rigorous proof on the following theorem about instability.

Theorem (Aw-Rascle, 2000)

When one of Riemann data is near the vacuum, that is, the light traffic, if and only if the solution presents instabilities.

- NOTE.

  Near the vacuum (i.e., the density is very low), the solution of Riemann's problem is very sensitive to the data.

  For example, any driver in everyday life can observe that there is ahead very light traffic of very slow drivers.

Ref. B.-C Huang, S.-W. Chou, J. M. Hong, and C.-C. Yen, Global transonic solutions of planetary atmospheres in a hydrodynamics region–hydrodynamic escape problem due to gravity and heat, SIAM, J. Math, Anal., Vol. 48. No. 6 (2016), pp. 4268–4310.

Theorem (Approximate Solution)

The approximate solution of the balance laws (multilane model) is

$$\begin{cases} \rho = \widetilde{\rho}, \\ v = (1-\theta)w(\mu, \widetilde{\rho}) + \theta \widetilde{v}, \end{cases}$$

where $\theta = e^{-t/\tau}$ *and* $w(\mu, \widetilde{\rho}) \equiv \mu w_1(\widetilde{\rho}) + (1-\mu) w_2(\widetilde{\rho})$.

By the theorem, it is clear that $v - \widetilde{v} = (1-\theta)(w(\mu, \widetilde{\rho}) - \widetilde{v})$.

A necessary and sufficient condition for the inequality

$$v \geq \widetilde{v}$$

to be true is that the following holds

$$\omega(\mu, \widetilde{\rho}) \geq \widetilde{v}.$$

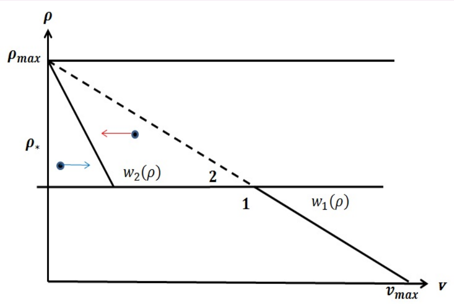
In addition, $v = \widetilde{v}$ if and only if $\omega(\mu, \widetilde{\rho}) = \widetilde{v}$.

Since we introduce the parameter μ, the convex combination speed $w(\mu, \widetilde{\rho})$ appears. So v is a convex combination of $\widetilde{v}$ and $w(\mu, \widetilde{\rho})$.

In particular,

- $(1,1,1)_{R,S} \Rightarrow \mu = 1 \Rightarrow \begin{cases} \lim_{\tau \to 0} v = w_1(\widetilde{\rho}), \\ w_1(\widetilde{\rho}) \geq \widetilde{v} \Rightarrow v \geq \widetilde{v}. \end{cases}$

- $(2,2,2)_{R,S} \Rightarrow \mu = 0 \Rightarrow \begin{cases} \lim_{\tau \to 0} v = w_2(\widetilde{\rho}), \\ \text{see the following figure.} \end{cases}$



It follows from $U_t + F(U)_x = G(U)$ and $U = \widetilde{U} + \overline{U}$ that

$$(\widetilde{U} + \overline{U})_t + F(\widetilde{U} + \overline{U})_x = G(\widetilde{U} + \overline{U}).$$

Performing the Taylor series expansion on F and G at $\widetilde{U}$ yields that

$$F(\widetilde{U} + \overline{U}) = F(\widetilde{U}) + DF(\widetilde{U})\overline{U} + \text{higher order terms},$$

and

$$G(\widetilde{U} + \overline{U}) = G(\widetilde{U}) + DG(\widetilde{U})\overline{U} + \text{higher order terms}$$

respectively. Removing higher order terms from two expansions and putting the remaining terms into the balance laws yields

$$\widetilde{U}_t + \overline{U}_t + F(\widetilde{U})_x + (DF(\widetilde{U})\overline{U}) = G(\widetilde{U}) + DG(\widetilde{U})\overline{U}.$$

This with the conservation laws $\widetilde{U}_t + +F(\widetilde{U})_x = 0$ gives

$$\overline{U}_t + (DF(\widetilde{U})\overline{U})_x = G(\widetilde{U}) + DG(\widetilde{U})\overline{U}.$$

We now have

$$\overline{U}_t + (DF(\widetilde{U})\overline{U})_x = G(\widetilde{U}) + DG(\widetilde{U})\overline{U}.$$

Apply the operator-splitting method, i.e., the term $\partial_x(DF(\widetilde{U})\overline{U})$ can be erased from the above equations:

$$\overline{U}_t = G(\widetilde{U}) + DG(\widetilde{U})\overline{U}$$

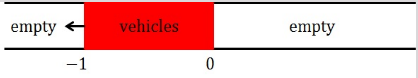
Hence we have the initial-value problem of ODE as follows.

$$\begin{cases} \overline{U}_t = G(\widetilde{U}) + DG(\widetilde{U})\overline{U}, \\ \overline{U}(x,0) = 0. \end{cases}$$

We introduce a new parameter $\mu = \mu(x)$, a function depending only on the space x so that the initial-value problem of ODE can be solved easily.

2. Diffusion

Example (Traffic light on the red)



Consider the stationary PW model with the initial condition

$$\rho(x,0) = \begin{cases} 1, \text{if} -1 \le x \le 0, \\ 0, \text{otherwise}, \end{cases} \quad \text{and } v(x,0) = 0,$$

and the boundary condition

$$\rho(0,t) = 1 \text{ and } v(0,t) = 0 \text{ for all } t \ge 0.$$

- The natural traffic evolution should be that nothing happens if the red light remains. However, there are some cars in the interval $(-\infty, -1)$. This shows that these cars travel with negative speed, which is completely unrealistic.

Before introducing the multilane model, we mention that both of the PW model and AR model are derived from the car-following model.

On the other hand, recall the acceleration equation from the LWR model:

$$a = v_x(-\rho v_e'(\rho)) = -\rho_x(v_e'(\rho))^2\rho.$$

If we choose $a = -\rho_x(v_e'(\rho))^2\rho$, this gives the PW model, and

if we choose $a = v_x(-\rho v_e'(\rho))$, this gives the AR model.