A boundary value problem for the Monge-Ampère equation

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   - A natural boundary condition
   - Applications

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   - A natural boundary condition
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2. **Global regularities**
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   - $W^{2,p}$ regularity
3. **Sketch of proof**
   - Uniform density
   - Uniform obliqueness

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   - Uniform density
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   - Regularity in dimension two
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subject to the natural boundary condition

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- This boundary value problem has many applications, such as in optimal transportation, minimal Lagrangian, & convex geometry.

APPLICATION IN OPTIMAL TRANSPORTATION

- Let $T : (\Omega, \rho) \to (\Omega^*, \rho^*)$ be the optimal mapping that minimising the total cost

$$\mathcal{C}(T) := \int_\Omega c(x, T(x))\rho(x)\, dx$$

among all measure preserving mappings, where ρ, ρ^* are two probability measures, and c is the cost function.

Application in optimal transportation

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- If $c(x, y) = \frac{1}{2}|x - y|^2$ (or equivalently $c(x, y) = x \cdot y$), the optimal mapping T is characterised by $T = Du$, where u is convex and satisfies the natural boundary value problem
$$\begin{cases} \det D^2 u &= \frac{\rho}{\rho^*(Du)} \quad \text{in } \Omega, \\ Du(\Omega) &= \Omega^*. \end{cases}$$

Application in geometry

- Let $\Omega, \Omega^* \subset \mathbb{R}^2$ be two smooth domains with equal area. Find an area-preserving diffeomorphism $F : \Omega \to \Omega^*$ such that the graph

$$\Sigma = \{(x, F(x)) : x \in \Omega\}$$

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- F is a.-p. implies that Σ is a Lagrangian surface in $(\mathbb{R}^4, \omega)$ with the symplectic form $\omega = dx_1 \wedge dx_2 - dy_1 \wedge dy_2$, where $dx_1 \wedge dx_2$ and $dy_1 \wedge dy_2$ are the standard area forms on Ω, Ω^*, respectively.

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- Σ is minimal implies that the Lagrangian angle β is constant. By choosing a proper β, one has $F = \nabla u$ and

$$\begin{cases} \det D^2 u & = 1 \quad \text{in } \Omega, \\ Du(\Omega) & = \Omega^*. \end{cases}$$

REMARKS

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- In the 1990's **Brenier** obtained the existence and uniqueness of a weak solution provided $|\partial\Omega| = |\partial\Omega^*| = 0$ and $\int_\Omega f = |\Omega^*|$.
- Our study focuses on the (global) regularity of the solution of

$$\begin{cases} \det D^2 u &= f \quad \text{in } \Omega, \\ Du(\Omega) &= \Omega^*. \end{cases}$$

Global regularity in Hölder space

Theorem (Delanoë, *Ann. Inst. H. Poincaré*, 1991)

*Assume that $\Omega, \Omega^* \subset \mathbb{R}^2$ are **uniformly convex** and smooth, then $f \in C^\infty(\overline{\Omega})$ implies that $u \in C^\infty(\overline{\Omega})$,*

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Theorem (Urbas, *J. Reine Angew. Math.*, 1997)

*Assume that $\Omega, \Omega^* \subset \mathbb{R}^n$ are **uniformly convex** and $\partial\Omega, \partial\Omega^* \in C^{2,1}$. Then $f \in C^{1,1}(\overline{\Omega})$ implies that $u \in C^{2,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$.*

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Theorem (Caffarelli, *Ann. Math.*, 1996)

Assume that $\partial\Omega, \partial\Omega^*$ are C^2 and ***uniformly convex****. Then $f \in C^\alpha(\overline{\Omega})$ with $\alpha \in (0,1)$ implies that $u \in C^{2,\alpha'}(\overline{\Omega})$ for some $0 < \alpha' < \alpha$.*

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Theorem (Caffarelli, *CPAM*, 1992)

Assume that Ω and Ω^* are bounded convex domains, and $f \geq 0$ satisfies the doubling condition. Then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

Global regularity in Sobolev space

Theorem (Chen-Figalli, *JFA*, 2017)

*Assume that Ω, Ω^* are C^2 smooth and **uniformly convex**. Then f is continuous implies that $u \in W^{2,p}(\overline{\Omega})$ for all $p \geq 1$.*

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- The global $W^{2,p}$ regularity for the Dirichlet problem was obtained by [Savin, *PAMS*, 2013].
- The interior $W^{2,p}$ (and also $C^{2,\alpha}$) estimate was proved by [Caffarelli, *Ann. Math.*, 1990].

Our main results

Theorem (Chen-L.-Wang, 2017, arXiv:1802.07518)

Assume that $\Omega, \Omega^* \subset \mathbb{R}^n$ *are* ***convex*** *with* $C^{1,1}$ *boundary. Then*
(i) $f \in C^\alpha(\overline{\Omega})$ *implies that* $u \in C^{2,\alpha}(\overline{\Omega})$ *for the same* $\alpha \in (0, 1)$.
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- We obtain the sharp boundary $C^{2,\alpha}$ regularity when $f \in C^\alpha(\overline{\Omega})$, for the same $\alpha \in (0,1)$.
- When f is Dini continuous, we also prove that $D^2 u \in C^0(\overline{\Omega})$.
- For Dirichlet problem, sharp boundary $C^{2,\alpha}$ estimates were obtain by [Trudinger-Wang, *Ann. Math.*, 2008] and [Savin, *JAMS*, 2013].

OUR STRATEGY

Our proof is based on delicate analysis on sub-level sets of solution

$$S_h(x_0) := \{x \in \Omega : u(x) < \ell_{x_0}(x) + h,\ \ell_{x_0} \text{ supports } u \text{ at } x_0\},$$

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near the boundary and uses various techniques on the Monge-Ampère equation. The main steps are:

1. Uniform density $(|\Omega \cap S_h^c(x)|/|S_h^c(x)| \geq \delta_0, \forall\, x \in \partial\Omega.)$
2. Tangential regularity $(u$ is $C^{1,\alpha}$ tangentially, $\forall\, \alpha \in (0,1).)$
3. Uniform obliqueness $(\langle \nu(x), \nu^*(Du(x))\rangle \geq \mu > 0, \forall\, x \in \partial\Omega.)$
4. Boundary regularity $(C^{2,\alpha}$ and $W^{2,p}.)$

STEP 1. UNIFORM DENSITY

LEMMA

Assume that Ω, Ω^* *are bounded convex domains with* $C^{1,1}$ *boundary, and that* $0 \in \partial\Omega$. *Then*

$$\frac{Vol(\Omega \cap S_h^c(0))}{Vol(S_h^c(0))} \geq \delta_0 > 0$$

for some positive constant δ_0, *independent of* u *and* h.

Step 1. Uniform density

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- The uniform density was proved by [Caffarelli, *Ann. Math.*, 1996], assuming that Ω is *polynomially convex*.
- We relax the polynomial convexity to the convexity of domains with $C^{1,1}$ boundary.

PROOF OF LEMMA

1. Assume that e_n is the inner normal of $\partial\Omega$ at 0. Let S'_h and $S'_{\Omega,h}$ be respectively the projections of $S^c_h(0)$ and $\Omega \cap S^c_h(0)$ on $\{x_n = 0\}$. Then, it suffices to prove

$$|S'_{\Omega,h}| \geq C|S'_h|. \tag{1}$$

Proof of Lemma

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2. For any unit vector $e \in \{x_n = 0\}$, denote

$$\lambda_e = \sup\{(x-y)\cdot e : x, y \in S'_{\Omega,h}\},$$
$$r_e = \sup\{t : te \in S'_h\}.$$

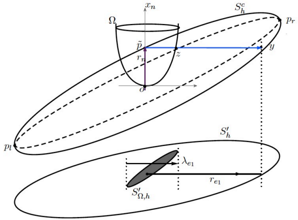
By induction, one can show that if

$$\frac{\lambda_e}{r_e} \geq C \quad \forall e \in \partial B_1(0) \cap \{x_n = 0\}, \tag{2}$$

then (1) holds.

Proof of Lemma

3. Assume λ_{e_1}/r_{e_1} is sufficiently small. Choose the most "right" point p_r and the most "left" point p_l on $\partial S_h^c(0)$. Then, for some $\delta \in (0,1)$



$$p_r \cdot e_1 \geq r_{e_1} > C z_n^{1/2} \geq C h^{\frac{1}{2(1+\delta)}};$$

$$|q_r - q_l| := |Du(p_r) - Du(p_l)| \leq C \frac{h}{p_r \cdot e_1}$$

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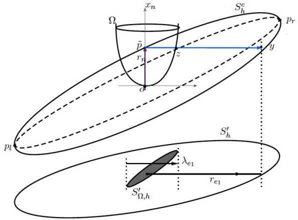

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4. Let p_0 be the minimum point of $u - \ell$ and $q_0 = Du(p_0) \in \overline{q_l q_r}$. Let $q^* = q_0 + \sigma e^* \in \partial\Omega^*$ s.t. $|q_0 - q^*| = dist(q_0, \partial\Omega^*)$. Then

$$\sigma \leq C|q_r - q_l|^2 \leq C h^{1+\frac{\delta}{1+\delta}}$$

$$d_{e^*} := \sup\{x \cdot e^* : x \in S_h^c(0)\} \approx h/\sigma \geq h^{-\frac{\delta}{1+\delta}} \to \infty$$

as $h \to 0$. Contradicts the strict convexity of u. □

Some corollaries

Corollary

Let T be a unimodular linear transform. If T normalises $S_h^c[u](0)$ s.t. $T\{S_h^c[u](0)\} \approx B_{Ch^{1/2}}$, then $T^* = (T^t)^{-1}$ normalises $S_h^c[v](0)$ s.t. $T^*\{S_h^c[v](0)\} \approx B_{Ch^{1/2}}$, where v is the dual potential function.

Corollary

For $h > 0$ is small, $\exists$ a constant C independent of u and h, s.t.

$$|x \cdot y| \le Ch, \quad \forall x \in S_h^c[u](0),\ y \in S_h^c[v](0),$$

and $\forall\ x \in \partial S_h^c[u](0)$, $\exists\ y \in \partial S_h^c[v](0)$ s.t.

$$x \cdot y \ge C^{-1} h.$$

Step 2. Tangential $C^{1,\alpha}$ regularity

Let $0 \in \partial\Omega$, locally $\partial\Omega = \{x_n = \rho(x')\}$ for some convex $\rho \in C^{1,1}$ and

$$\rho(0) = 0, \quad D\rho(0) = 0.$$

Let f be a positive continuous function. It suffices to show

Lemma

$\forall\ \alpha \in (0,1)$, $\exists$ a small constant $C = C_\alpha > 0$, independent of h, s.t.

$$S_h^c(0) \cap \{x_n = 0\} \supset B_{C_\alpha h^{1/(1+\alpha)}}(0) \cap \{x_n = 0\}.$$

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- Note that if $\partial\Omega \in C^2$ and uniformly convex, it will be quadratic after blowing-up as in [Caffarelli, *Ann. Math.*, 1996].
- In our case, we alternatively control the "slope" of S_h^c along the blowing-up process.

Strategy of the proof

1. If $C^{1,\alpha}$ fails in e_1 direction, i.e. S_h^c is too "narrow" in e_1 direction, then after normalisation $\mathcal{T}$, $\mathcal{T}(\Omega)$ tends to be flat in e_1 direction.

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2. Denote D_h by erasing the dependence on x_1 in $\mathcal{T}(S_h^c \cap \Omega)$. Approximate u by a solution w of

$$\begin{cases} \det D^2 w = \chi_{D_h} & \text{in } \mathcal{T}(S_h^c), \\ w = u & \text{on } \partial\mathcal{T}(S_h^c). \end{cases}$$

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3. By Pogorelov's interior C^2 estimate, w is $C^{1,1}$ in e_1 direction.
4. Note that $Vol\{D_h - \mathcal{T}(S_h^c \cap \Omega)\} = o(h)$. By the maximum principle, $|w - u| \to 0$ as $h \to 0$. Changing back to the second variation of u in e_1, we derive a contradiction. □

Step 3. Uniform obliqueness

Lemma

Assume that $\partial\Omega, \partial\Omega^* \in C^{1,1}$ *are convex,* $f \in C^0$ *is positive. Let* $0 \in \partial\Omega$ *and* $Du(0) = 0 \in \partial\Omega^*$*. Then* $\exists$ *a positive constant* μ *s.t.*

$$\langle \nu(0), \nu^*(0) \rangle \geq \mu > 0,$$

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- Uniform obliqueness is a key ingredient in obtaining the boundary regularity, and is also necessary for D^2u to be bounded.
- Previously, both the uniform convexity and smoothness of $\partial\Omega, \partial\Omega^*$ play a critical role, as defining functions provide natural barriers.
- We relax these assumptions and give a completely different proof.

Proof of obliqueness

For convenience, we demonstrate our main ideas in dimension two.

1. Suppose to the contrary that $\nu(0) = e_2$ and $\nu^*(0) = e_1$, we have

$$u_1 = u_{x_1} > 0 \text{ in } \Omega,$$
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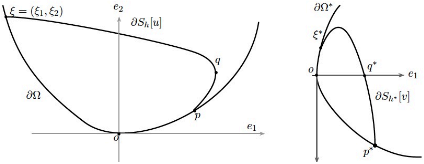

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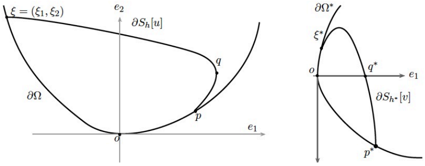

Let q, ξ be the most "right", "left" points on $\partial S_h(0) \cap \overline{\Omega}$, respectively.

2. S_h is almost "balanced", in the sense that, for all $h > 0$ small

$$q_1 \geq \delta_0 |\xi_1|, \quad q = (q_1, q_2), \xi = (\xi_1, \xi_2),$$

where $\delta_0 > 0$ is a constant independent of h.

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3. Assume locally $\partial\Omega = \{x_2 = \rho(x_1)\}$ with $\rho(t) \le |t|^{1+\alpha}$ for some $\alpha > 0$. For $t > 0$, denote

$$\underline{u}(t) := \inf\{u(t, x_2) : x_2 \ge \rho(t)\},$$
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5. The above asymptotic behaviour is preserved along a proper blowing-up sequence, and the limit profile is

$$\det D^2 u = f(0) \quad \text{in } \Omega_0 = \{x_2 > \rho_0(x_1)\},$$

for some $\rho_0 \geq 0$ convex, satisfing $\rho_0(t) \leq Ct^{1+\alpha}$ for $t > 0$ small.

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10. It implies that $\underline{w}(t) \equiv 0$ for all $t > 0$ small, which contradicts with the fact that $u(x) > 0, u_1(x) > 0$ for all $x \neq 0$. □

Step 4. Global estimates

Lemma

Assume that $\partial\Omega, \partial\Omega^* \in C^{1,1}$ *are convex,* $f \in C^0$ *is positive. Then* $u \in C^{1,1-\varepsilon}(\overline{\Omega})$, *for any small* $\varepsilon > 0$, .

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- Let a_h be the smallest number s.t. $D^+_{h,a_h} \subset \Omega$. Denote D^+_{h,a_h} by D^+_h.
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- It suffices to show D_h is close to a ball of radius $h^{1/2}$, namely

$$B_{C^{-1}h^{\frac{1}{2}+\varepsilon}}(x_h) \subset D_h \subset B_{Ch^{\frac{1}{2}-\varepsilon}}(x_h),$$

for any given small $\varepsilon > 0$, where the centre $x_h = a_h e_n$.

GLOBAL $W^{2,p}$ ESTIMATE

To prove the global $W^{2,p}$ estimate, we have following observations:

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4. Let $d = 2^{-k}$, $k = 0, 1, 2, \cdots$. By adding over k we obtain
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5. $\forall p \ge 1$, as $3\varepsilon p < 1/4$, the above series is convergent. □

GLOBAL $C^{2,\alpha}$ ESTIMATE

- The global $C^{2,\alpha'}$ estimate for some $\alpha' \in (0, \alpha)$ was established in [Caffarelli, *Ann. Math.*, 1996]. We found that the exponent α' can be improved to the same α, (namely $f \in C^{\alpha}(\overline{\Omega}) \Longrightarrow u \in C^{2,\alpha}(\overline{\Omega})$).
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- In the following we adopt the techniques from [Wang, *Chinese Ann. Math.*, 2006] and [Jian-Wang, *SIAM J. Math. Anal.*, 2007].

1. **Approximation sequence:** Let w be the solution of
$$\det D^2 w = 1 \text{ in } D_h, \quad w = h \text{ on } \partial D_h.$$
2. **Maximum principle:** Let $|f - 1| \le h^\delta$ for some $\delta \in (0, 1/2)$, then
$$|u - w| \le Ch^{1+\delta} \quad \text{in } D_h \cap \Omega.$$
3. **Continuity estimate:** Let $h_k = 4^{-k}$, $k = 1, 2, \cdots$. Approximate u by solving $\det D^2 u_k = f_k$ in D_k, $u_k = h_k$ on ∂D_k, we can obtain
$$|D^2 u(z) - D^2 u(0)| \le C|z|^\alpha. \qquad \square$$

Further remarks

- Our argument also implies that if f is Dini continuous, that is if

$$\int_0^1 \frac{\omega_f(t)}{t} dt < \infty, \quad \omega(t) = \sup\{f(x) - f(y) : |x - y| < t\},$$

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- We can further reduce the $C^{1,1}$ smoothness assumption on the boundaries $\partial\Omega, \partial\Omega^*$ to be $C^{1,1-\theta}$ for some small $\theta > 0$ depending on the global $C^{1,\delta}$ regularity of u. (see the next slide)

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- From [Ma-Trudinger-Wang, *ARMA*, 2005], it's known that $\forall f > 0$ smooth, the convexity of domains is necessary for $u \in C^1(\overline{\Omega})$. However, for a fixed $f > 0$, by our results and a perturbation argument, we can show that u is smooth up to the boundary, if the domains are smooth perturbations of convex ones, even though they are not convex themselves. (see the next, next slide)

Recent results in dimension two

Very recently, by an iteration argument we obtain the following regularity results in dimension two.

Theorem (Chen-L.-Wang, 2018, arXiv:1806.09482)

Assume that Ω, Ω^* are two bounded convex domains in $\mathbb{R}^2$.

- *If $\partial\Omega, \partial\Omega^* \in C^{1,\alpha}$, and $f \in C^\alpha(\overline{\Omega})$ is positive, for some $\alpha \in (0,1)$. Then $\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$ for the same α.*
- *If $\partial\Omega, \partial\Omega^* \in C^{1,\varepsilon}$ for some $\varepsilon > 0$, and $f \in C^0(\overline{\Omega})$ is positive. Then $\|u\|_{W^{2,p}(\overline{\Omega})} \leq C$ for all $p \geq 1$.*
- *With no further smoothness assumption on domains, and $\lambda^{-1} < f < \lambda$ for some $\lambda > 0$. Then $\|u\|_{W^{2,1+\epsilon}(\overline{\Omega})} \leq C$, where $\epsilon, C > 0$ are universal constants depending only on λ and the domains.*

Recent results in non-convex domains

- Given a bounded domain $\Lambda \subset \mathbb{R}^n$, we say Λ is δ-*close* to Ω in $C^{1,1}$ norm, if $\exists$ a bijective mapping $\Phi : \Omega \to \Lambda$ s.t. $\Phi \in C^{1,1}(\overline{\Omega})$ and

$$\|\Phi - Id\|_{C^{1,1}(\overline{\Omega})} \leq \delta.$$

Theorem (Chen-L.-Wang, *SCIENCE CHINA Mathematics*, 2019)

Let Λ, Λ^* be $C^{1,1}$ domains that are δ-close to convex domains Ω, Ω^* in $C^{1,1}$ norm, respectively. Suppose that $f \in C^\alpha(\overline{\Lambda})$, for some $\alpha \in (0,1)$. Then, $\exists$ a small constant $\delta_0 > 0$ depending only on Ω, Ω^* and $\|f\|_{C^\alpha(\overline{\Lambda})}$, s.t. the solution $u \in C^{2,\alpha}(\overline{\Lambda})$, provided $\delta < \delta_0$.

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▶ Similarly, we also have the global $W^{2,p}$ estimate when $f \in C^0(\overline{\Lambda})$, provided the perturbation δ_0 is small enough.

▶ Interesting application to *Wolfson's problem*: it allows the curvature $\kappa > -\epsilon_0$, $\Longrightarrow \exists$ the minimal Lagrangian diffeomorphism.

FIGURE: A picture of sunrise at Gold Coast, Australia

Thank you !