Spatial behavior of the solution to the Boltzmann equation with hard potentials

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Outline

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Kinetic theory

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$$F(x,t,\xi),\ \text{density distribution function of gas particle}$$

$x \in \mathbb{R}^3$ space, t time, $\xi \in \mathbb{R}^3$ microscopic velocity.

- Macroscopic variables

$$\begin{cases} \rho(x,t) = \int_{\mathbb{R}^3} F(x,t,\xi)d\xi, \ \text{density}, \\ \rho(x,t)v(x,t) = \int_{\mathbb{R}^3} \xi F(x,t,\xi)d\xi, \ \text{momentum}. \end{cases}$$

$$\begin{cases} \rho(x,t)E(x,t) = \int_{\mathbb{R}^3} \frac{|\xi|^2}{2} F(x,t,\xi)d\xi, \ \text{total energy}, \\ \rho(x,t)e(x,t) = \int_{\mathbb{R}^3} \frac{|\xi-v|^2}{2} F(x,t,\xi)d\xi, \ \text{internal energy}, \\ \rho E = \rho e + \frac{1}{2}\rho|v|^2. \end{cases}$$

$$\begin{cases} p_{ij}(x,t) = \int_{\mathbb{R}^3} (\xi_i - v_i)(\xi_j - v_j) F(x,t,\xi)d\xi, \ \text{stress tensor}, \\ p(x,t) = \frac{p_{11}+p_{22}+p_{33}}{3} = \int_{\mathbb{R}^3} \frac{|\xi-v|^2}{3} F(x,t,\xi)d\xi, \ \text{pressure}, \\ q_i(x,t) = \int_{\mathbb{R}^3} (\xi_i - v_i)\frac{|\xi-v|^2}{2} F(x,t,\xi)d\xi, \ \text{heat flux}. \end{cases}$$

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$$\partial_t F + \xi \cdot \frac{\partial F}{\partial x} = Q(F,F)$$

- Transport: $\partial_t F + \xi \cdot \frac{\partial F}{\partial x}$
- Collision operator:

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*)\right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

$$\begin{cases} \xi' = \xi + [(\xi_* - \xi)\cdot\Omega]\Omega, \\ \xi'_* = \xi_* - [(\xi_* - \xi)\cdot\Omega]\Omega. \end{cases}$$

- Hard sphere: $B(|\xi - \xi_*|, \theta) = |(\xi_* - \xi)\cdot\Omega| = |\xi_* - \xi| \cos\theta$
- Hard potential: $0 < \gamma < 1$

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- Collision invariants and conservation laws

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ \frac{1}{2}|\xi|^2 \end{pmatrix} [\partial_t F + \xi \cdot \nabla_x F] \, d\xi = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ \frac{1}{2}|\xi|^2 \end{pmatrix} Q(F, F) d\xi = 0$$

- Conservation laws for the macroscopic variables:

$$\partial_t \rho + \nabla_x \cdot (\rho v) = 0, \text{ mass},$$

$$\partial_t (\rho v) + \nabla_x \cdot (\rho v \otimes v + P) = 0, \text{ momemtum},$$

$$\partial_t (\rho E) + \nabla_x \cdot (\rho v E + P v + q) = 0, \text{ energy}.$$

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- H-theorem

$$H := \int_{\mathbb{R}^3} F \log F d\xi, \quad \vec{H} := \int_{\mathbb{R}^3} \xi F \log F d\xi.$$

$$\begin{aligned} \partial_t H + \nabla_x \cdot \vec{H} &= \int_{\mathbb{R}^3} Q(F, F) \log F d\xi \\ &= \frac{1}{4} \int_{\mathbb{R}^6} \int_{\mathbf{S}^2_+} \log \frac{FF_*}{F'F'_*} \left[F'F'_* - FF_* \right] B d\Omega d\xi_* d\xi \leq 0, \end{aligned}$$

$= 0$ if and only if

$$F(x, t, \xi) = \frac{\rho(x, t)}{(2\pi R\theta(x, t))^{3/2}} e^{-\frac{|\xi - v(x,t)|^2}{2R\theta(x,t)}} =: M_{[\rho, v, \theta]},$$

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Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Macro-Micro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to \text{Ker } L$
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = LP_0 = 0, \quad P_1 L = LP_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = LP_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$
$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t(P_0 f) + \partial_x(P_0 \xi_1 P_0 f) = -\partial_x(P_0 \xi_1 P_1 f) = -\partial_x^2\Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t(P_0 f) + \partial_x(P_0\xi_1 P_0 f) = \partial_x^2\Big[\underbrace{P_0\xi_1(-L)^{-1}P_1\xi_1}_{\text{viscosity matrix}} P_0 f\Big]$$

- Note $P_0 f$ is 3 dimensional, in terms of the basis, the above equation is a viscous conservation law system.

- Diagonalize the system gives 3 eigenvalues

$$\Big\{\lambda_1 = -\sqrt{5/3},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{5/3}\Big\}$$

$\sqrt{5/3}$ sound speed, 0 is the background velocity. The numerical values are due to our linearization $M = (2\pi)^{-3/2}\exp(-|\xi|^2/2)$. In general

$$\{\lambda_1 = u - c,\ \lambda_2 = u,\ \lambda_3 = u + c\}$$

Outline

Long-Short wave decomposition

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f = Lf \,, \quad (t,x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \,, \\ f(0,x,\xi) = f_0(x,\xi) \end{cases}$$

- f_0 cpt. supp. in x and polynomial ξ weight $\langle \xi \rangle^\beta$ ($\beta = (3/2)^+$)

$$f(t,x,\xi) = \int_{\mathbb{R}^3} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0(\eta,\xi)\, d\eta$$

$$f_L(t,x,\xi) = \int_{|\eta|\le 1} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0(\eta,\xi)\, d\eta$$

$$f_S(t,x,\xi) = \int_{|\eta|> 1} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0(\eta,\xi)\, d\eta$$

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Fluid structure $0 \leq \gamma < 1$

Let $v = \sqrt{5/3}$ be the sound speed associated with normalized global Maxwellian, and any given Mach number $\mathbb{M} > 1$, there exists positive constant C such that for $|x| \leq (\mathbb{M}+1)\, vt$,

$$
\begin{aligned}
|f_L|_{L^2_\xi} \leq C \Bigg[& (1+t)^{-2} \left(1 + \frac{(|x| - vt)^2}{1+t}\right)^{-N} \\
& + (1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t}\right)^{-N} \\
& + \mathbf{1}_{|x| \leq ct} (1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t}\right)^{-3/2} \Bigg] \|f_0\|_{L^1_x L^2_\xi} .
\end{aligned}
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\|f_S\|_{L^2} \lesssim e^{-ct} \|f_0\|_{L^2}
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Wave-Remainder decomposition

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0,x,\xi) = f_0(x,\xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0,x,\xi) = 0 \end{cases}$$

We can define the wave part and remainder part:

$$W^{(9)} = \sum_{j=0}^{9} h^{(j)}\,, \quad \mathcal{R}^{(9)} = f - W^{(9)}$$

Note that $\mathcal{R}^{(9)}$ solves the equation

$$\begin{cases} \partial_t \mathcal{R}^{(9)} + \xi \cdot \nabla_x \mathcal{R}^{(9)} = L\mathcal{R}^{(9)} + K h^{(9)}\,, \\ \mathcal{R}^{(9)}(0,x,\xi) = 0 \end{cases}$$

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Wave-Remainder decomposition

- Damped transport operator: $h(t) = \mathbb{S}^t h_0$
$\partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0$, with $h(0,x,\xi) = h_0$

Let $\alpha \geq 0$. Then for $0 \leq \gamma < 1$ and any positive number σ,

$$\left|\mathbb{S}^t \langle \xi \rangle^{-\alpha} h_0\right|_{L^\infty_\xi} \leq C \sup_y e^{-\frac{\nu_0}{3} t} (1 + |x-y|)^{-\frac{\alpha}{1-\gamma}} \frac{t^{-\sigma}}{t^{-\sigma} + (1+|x-y|)^{\frac{\sigma\gamma}{1-\gamma}}} |h_0(y,\cdot)|_{L^\infty_\xi} .$$

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- $0 < \gamma < 1$, with polynomial weight $\langle \xi \rangle^{\alpha}$

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$$\frac{t^{-\sigma}}{t^{-\sigma} + \left(1 + |x - y|\right)^{\frac{\sigma\gamma}{1-\gamma}}} \left| h_0 \left(y, \cdot\right) \right|_{L^\infty_\xi(\langle \xi \rangle^{\alpha})} .$$

- $0 < \gamma < 1$, with exponential weight $e^{\alpha |\xi|^p}$ (Lin-Wang-Wu, JSP)

$$\left| \mathbb{S}^t h_0 \right|_{L^\infty_\xi} \le \sup_y e^{-c_0 \left(t + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}} \right)} \left| h_0 \left(y, \cdot\right) \right|_{L^\infty_\xi\left(e^{\alpha |\xi|^p}\right)}$$

- $\gamma = 1$, (Liu-Yu, CPAM)

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- Let $j \geq 1$, $\beta = (3/2)^{+}$, $0 \leq \gamma < 1$,

$$\left| h^{(j)} \right|_{L^\infty_\xi} \leq e^{-c_0 t} \langle x \rangle^{-\frac{\gamma+\alpha_j+\beta_j}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$$

where $\alpha_j = \frac{j(2-\gamma)}{j+1}$, $\beta_j = \frac{\beta}{j+1}$.

- Key fact: $|Kg(\xi)| \leq \langle \xi \rangle^{-(2-\gamma)} |g|_{L^\infty_\xi}$.

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- Why $W^{(9)}$?
- How to estimate $\mathcal{R}^{(9)}$

- Regularization estimate, Weighted energy estimate

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- $f = W^{(9)} + \mathcal{R}^{(9)}$, we have accurate description of $W^{(9)}$.
- (Weighted energy estimate for $\mathcal{R}^{(9)}$, $0 < \gamma < 1$)

$$w(t,x,\xi) = 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{2}{1-\gamma}}(1-\chi) + \left[(1-\chi)\left[\delta\left(\langle x\rangle - Mt\right)\right]\langle\xi\rangle_D^{\gamma+1} + 3\langle\xi\rangle_D^2\right]\chi.$$

where $\langle\xi\rangle_D = (D + |\xi|^2)^{1/2}, \chi = \chi\left(\frac{\delta(\langle x\rangle - Mt)}{\langle\xi\rangle_D^{1-\gamma}}\right)$ and $\mu(x,\xi) = w(0,x,\xi)$.

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$$\partial_t u + \xi\cdot\nabla_x u - (\partial_t w + \xi\cdot\nabla_x w)w^{-1}u - wL\left(w^{-1}u\right) = wKh^{(9)}.$$

- $\int_{\mathbb{R}^3}\left\langle u, wLw^{-1}u\right\rangle_\xi dx \approx \int_{\mathbb{R}^3}\left\langle u, Lu\right\rangle_\xi dx + \frac{1}{D}P(\|\mathbb{P}_0 f\|_{L^2})$
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Linear Theory

Theorem

Let f be a solution to the Boltzmann equation with initial data compactly supported in the x-variable and bounded in $L^\infty_{\xi,\beta}$ space, $\beta = (3/2)^+$, then $\left|W^{(9)}\right|_{L^\infty_\xi} \le e^{-c_0 t}\langle x\rangle^{-\frac{3/2}{1-\gamma}}\|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$.

(a) $|x| < Mt$

$$\left|\mathcal{R}^{(9)}\right|_{L^2_\xi} \le C_N \left[\begin{array}{l} (1+t)^{-2}\left(1+\frac{(|x|-\mathbf{v}t)^2}{1+t}\right)^{-N} \\ +(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N} \\ +\mathbf{1}_{\{|x|\le \mathbf{v}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2} \end{array}\right]\|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}.$$

(b) $|x| > Mt$

$$\left|\mathcal{R}^{(9)}\right|_{L^2_\xi} \le (1+t)^{-N/4}(t+x)^{-\frac{3/2}{1-\gamma}}\|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}.$$

Linear theory

Theorem (L^∞_ξ estimate)

Let f be a solution to the Boltzmann equation with initial data compactly supported in the x-variable and bounded in $L^\infty_{\xi,\beta}$ space, $\beta = (3/2)^+$, then

$$|f|_{L^\infty_\xi} \le C_N \left[\begin{array}{l} (1+t)^{-2}\left(1+\frac{(|x|-\mathbf{v}t)^2}{1+t}\right)^{-\frac{3/2}{1-\gamma}} \\ +(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N} \\ +\mathbf{1}_{\{|x|\le \mathbf{v}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2} \\ +(1+t)^{-3/2}(t+x)^{-\frac{3/2}{1-\gamma}} \end{array} \right] \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}} .$$

Outline

Nonlinear theorem

- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f)\,, \quad f(0, x, \xi) = f_0$ small
- $f_0 \in H^N_x L^2_\xi(\langle\xi\rangle^p)$, $p \geq 2$, $N \geq 4$.
- Existence, uniqueness and optimal time decay: Guo, Yang-Yu ....

Define
$\mathcal{E}(f)(t) = \sum_{|\alpha|\leq N} \|\partial^\alpha_x f\|^2_{L^2} + \sum_{|\alpha|\leq N} \| \langle\xi\rangle^p \, \partial^\alpha_x f\|^2_{L^2}\,.$
$\mathcal{D}(f)(t) = \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x \mathrm{P}_0 f\|^2_{L^2} + \sum_{|\alpha|\leq N} \| \langle\xi\rangle^p \, \partial^\alpha_x \mathrm{P}_1 f\|^2_{L^2_\sigma}\,.$
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If we assume $\mathcal{E}(f)(0) \leq \eta$, then

- $\mathcal{E}(f)(t) + \int_0^t \mathcal{D}(f)(s) ds \leq \mathcal{E}(f)(0)\,.$
- $\|f\|^2_{L^2} = \|\mathrm{P}_0 f\|^2_{L^2} + \|\mathrm{P}_1 f\|^2_{L^2} \leq \eta(1+t)^{-3/2}\,,$
- $E(f)(t) \leq \eta(1+t)^{-5/2}\,.$

Nonlinear theorem

- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$ small
- $f_0 \in H^N_x L^2_\xi(\langle\xi\rangle^p)$, $p \geq 2$, $N \geq 4$.
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Let $g = wf$, then g solves the equation

$$\partial_t g + \xi \cdot \nabla_x g - (\partial_t w + \xi \cdot \nabla_x w) w^{-1} g - wL\left(w^{-1} g\right) = w\Gamma(w^{-1} g, f)\,.$$

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$$\leq D^{-2} P(|\mathrm{P}_1 u|_{L^2_\sigma}, |\mathrm{P}_0 u|_{L^2_\xi}, \mathcal{D}(f), E(f), \|f\|_{L^2})$$

Theorem (Energy estimate for fully nonlinear equation)

Let f be a solution to the Boltzmann equation with small initial data $f_0 \in H^N_x L^2_\xi(\langle \xi \rangle^p)$, $N \geq 4$, if $|x| > Mt$ for some M large, then

$$|f|_{L^2_\xi} \leq C(1 + |x|)^{-\frac{p}{1-\gamma}} \|f_0\|_{H^4_x L^2_\xi(\langle \xi \rangle^p)}\,.$$

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THANK YOU