Existence and uniqueness for nonlocal and crystalline mean curvature flows

Massimiliano Morini
University of Parma

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Plan of the talk

Two classical ''weak'' methods

Nonlocal motions: a unified theory

Crystalline flows: existence and uniqueness

Outline

Two classical ''weak'' methods

Nonlocal motions: a unified theory

Crystalline flows: existence and uniqueness

A classical geometric evolution

Motion by mean curvature: $t \mapsto E_t \subset \mathbb{R}^d$

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- Singularities may appear even from smooth initial data:

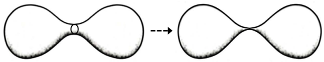

Figure: An example of pinching singularity (Grayson '89).

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Figure: An example of pinching singularity (Grayson '89).

Question: How to define a global-in-time solution? How to define a solution starting from irregular initial sets?

The level set approach

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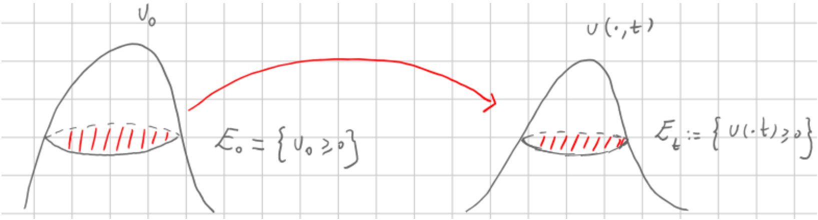

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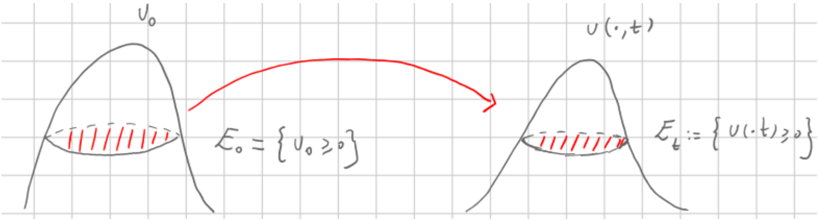

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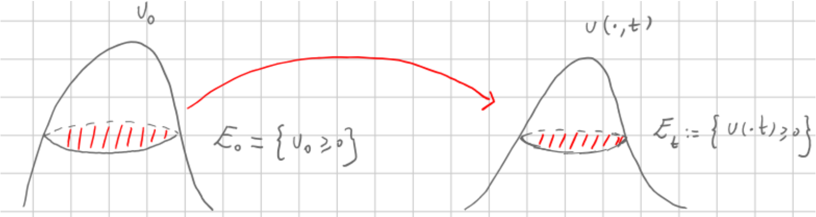

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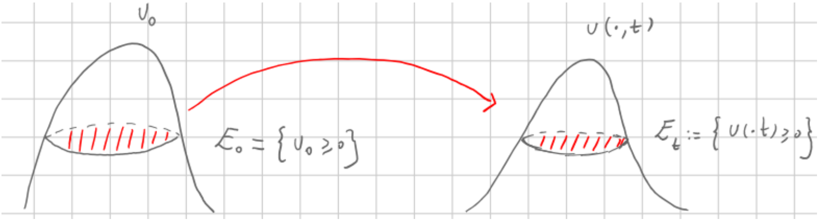

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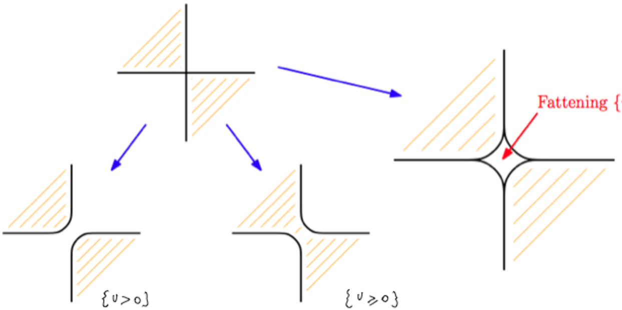
- Global existence and uniqueness for (LS) by Evans-Spruck (1991) and Chen-Giga-Goto (1991) with the machinery of viscosity solutions.

Non uniqueness by fattening

If one fixes the level set, uniqueness can only hold up to fattening:



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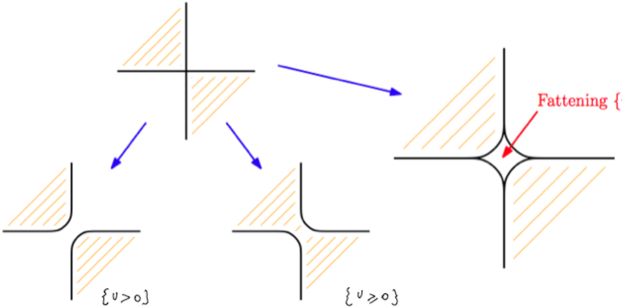

Generic Uniqueness : For all but countably many s, no fattening occurs and the evolution E_s is unique.

The ATW minimizing movements approach

Minimizing movements: $E_{n-1} \mapsto E_n$

$$\min \left(\mathrm{Per}(F) + \frac{1}{h} \int_{F \Delta E_{n-1}} \mathrm{dist}(x, \partial E_{n-1})\, dx \right) \qquad \text{(ATW)}$$

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Question: Does uniqueness of flat flows (up to fattening) hold also in the general anisotropic, possibly crystalline, case?

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Generalized (nonlocal) perimeters

Chambolle-M.-Ponsiglione, *Arch. Ration. Mech. Anal.* (2015)

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- $J(\emptyset) = J(\mathbb{R}^d) = 0$
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J is submodular $\iff$ $\widetilde{J}$ is convex (Chambolle, Giacomini, Lussardi 2010)

Generalized (nonlocal) curvatures

Definition

We say that $\kappa(\cdot, E)$ *is the curvature of* ∂E *w.r.t.* J *if for any smooth* $(\Phi_\varepsilon)_\varepsilon$*, with* $\Phi_0 = Id$*, setting* $X := \frac{\partial \Phi_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0}$*, one has*

$$\frac{d}{d\varepsilon} J\big(\Phi_\varepsilon(E)\big)_{|\varepsilon=0} = \int_{\partial E} \kappa(x, E)\, X(x) \cdot \nu^E(x) d\mathcal{H}^{N-1}(x).$$

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Standing assumptions:

- Existence: $\kappa(\cdot, E)$ is defined for all E of class C^2
- Continuity: If $E_n \to E$ in C^2 and $x_n \in \partial E_n \to x \in \partial E$, then $\kappa(x_n, E_n) \to \kappa(x, E)$
- Non degeneracy: $\inf_{\rho>0} \min_{x \in \partial B_\rho} \kappa(x, B_\rho) > -\infty$

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Lemma (Monotonicity)

Let $E, F \in C^2$ with $E \subseteq F$ and let $x \in \partial F \cap \partial E$. Then

$$\kappa(x, F) \leq \kappa(x, E).$$

Level set formulation of nonlocal geometric flows

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- Weak formulation: The curvature κ is defined only on regular sets. We consider viscosity solutions.

A level-by-level generalized ATW scheme

For any fixed time step $h > 0$, let $T_h E$ be the minimal solution to

$$\min_{F \subset \mathbb{R}^d} \left\{ J(F) \; + \; \frac{1}{h} \int_{F \triangle E} \mathrm{dist}(x, \partial E) \, dx \right\}$$

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$E \subseteq E' \Longrightarrow T_h E \subseteq T_h E'$ *and* $\operatorname{dist}\left(T_h E, (T_h E')^c\right) \geq \operatorname{dist}(E, E'^c)$

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$$T_h u(x) \; := \; \sup\{s \; : \; x \in T_h\{u \geq s\}\} \, .$$

- Let $u_0 \in BUC(\mathbb{R}^d)$, constant outside a compact set. We define

$$u_h(x,t) \; := \; (T_h)^{[\frac{t}{h}]} u_0 \, .$$

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It can be shown that, up to subsequences, $u_h \to u$ uniformly on compact sets

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Theorem (Chambolle-M.-Ponsiglione, ARMA 2015)

The limiting function u is a viscosity solution of

$$\begin{cases} u_t(x,t) + |Du(x,t)|\kappa(x, \{y :\ u(y,t) \geq u(x,t)\}) \ = \ 0 \\ u(0,\cdot) = u_0. \end{cases}$$

Moreover , if κ *is* "uniformly continuous" *with respect to* C^2 *-convergence of sets, then the level set flow is unique, it obeys the comparison principle, and the set flow* $t \mapsto \{x :\ u(x,t) \geq s\}$ *depends only on* $\{u^0 \geq s\}$.

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Barchiesi, Kang, Lee, Morini, Ponsiglione, Multiscale Model. Simul. (2010)

Limitations of the theory

Consider a norm ϕ and the corresponding anisotropic perimeter

$$P_\phi(E) = \int_{\partial E} \phi(\nu^E)\, d\mathcal{H}^{d-1}$$

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- If ϕ is smooth (and $m(\nu) \equiv 1$) we apply previous theory
- If ϕ is non-smooth (e.g. crystalline), then the Cahn-Hoffmann field $\nabla\phi(\nu^E)$ and hence κ_ϕ^E are not well defined in a classical way.

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- If ϕ is smooth (and $m(\nu) \equiv 1$) we apply previous theory
- If ϕ is non-smooth (e.g. crystalline), then the Cahn-Hoffmann field $\nabla\phi(\nu^E)$ and hence κ_ϕ^E are not well defined in a classical way. The previous theory does not apply

Outline

Two classical ''weak'' methods

Nonlocal motions: a unified theory

Crystalline flows: existence and uniqueness

The crystalline case

The unit ball B_ϕ

The Wulff shape W_ϕ

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- The curvature becomes nonlocal!

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- The case $d \geq 3$: investigated by many authors, only partial results were available prior to ours:
  - Convex initial data: Bellettini, Caselles, Chambolle & Novaga (2008)
  - Polyhedral initial data: Giga, Gurtin & Matias (1998)
  - the well-posedness and the validity of a comparison principle in the general case has been a long-standing open problem as well as the uniqueness of the crystalline flat flow

Recent developments

Chambolle-M.-Ponsiglione 2016

Let ϕ be any (possibly crystalline) anisotropy. Then, the anisotropic mean curvature equation

$$V = -\phi(\nu)\,\kappa_\phi$$

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- The result holds for the "natural" mobility $m = \phi$

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- Analogously, setting $d^c(\cdot, t) := \mathrm{dist}(\cdot, E^c(t))$, we have

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Our new weak formulation of $V = -\phi(\nu)\,\kappa_\phi$

Definition

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Let $A := (A(t))_{t\geq 0} \subseteq \mathbb{R}^{\mathbb{N}} \times [0, +\infty)$ *be a (relatively) open tube. We say that* A *is a* *weak subflow* *if* $\mathbb{R}^{\mathbb{N}} \times [0, +\infty) \setminus A$ *is weak superflow.*

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- Existence: via minimizing movements

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Let $E(0) \subset A(0)$ and let $\Delta > 0$ be the distance of their boundaries. Let E be a weak superflow, and A a weak subflow.

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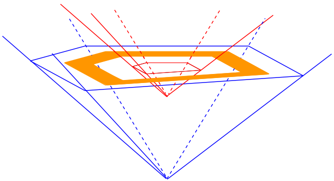

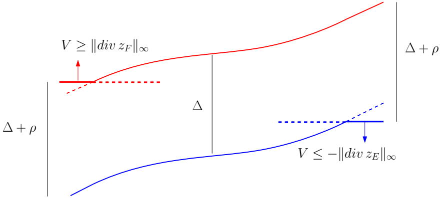

Parabolic maximum principle: In a strip $S \subset\subset A \setminus E$, we want to prove that $\Delta(t) \geq \Delta$ (at least for short time).

Distances are "rigid": $\Delta(t) \geq \Delta$ everywhere

Iteration: $\Delta(t) \geq \Delta$ for all times (before T^*).

Existence and uniqueness for $V = -\phi(\nu)\,\kappa_\phi$

Theorem (Chambolle-M.-Ponsiglione, CPAM 2016)

Let ϕ be any anisotropy and u^0 be a uniformly continuous function in $\mathbb{R}^N$. Then, for all but countably many $s \in \mathbb{R}$ the minimizing movements scheme starting from $E_s^0 := \{u^0 \geq s\}$ converge to the unique weak solution E_s of $V = -\phi(\nu)\,\kappa_\phi$, with initial datum E_s^0.

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After our preprint appeared, Giga-Pozar (preprint 2016): viscosity approach in three-dimensions for

$$V = -m(\nu)(\kappa_\phi + 1)\,,$$

for bounded initial sets and when ϕ is purely crystalline.

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Definition (ϕ-regular mobilities)

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The techniques of Chambolle-M.-Ponsiglione can be pushed to treat $V = -m(\nu^{E(t)})(\kappa_\phi^{E(t)} + g(x,t))$, when m is ϕ-regular and g is bounded forcing term with spatial Lipschitz continuity

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Note: the equations $V = -\kappa_\phi$ or $V = -(\kappa_\phi + 1)$, with ϕ crystalline, are not covered

General mobilities

Theorem (Chambolle-M.-Novaga-Ponsiglione, to appear on JAMS)

For any ϕ and m there exists a unique level set flow $u^{\phi,m}$ corresponding to $V = -m(\nu)(\kappa_\phi + g)$, with initial datum u^0 .

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- Idea: Let $m_n \to m$, where m_n is ϕ-regular. Then, by delicate stability estimates on the ATW scheme one can show that the corresponding $\{u^{\phi,m_n}\}$ admit a unique limit.
- The long-standing problem of the well-posedness of crystalline flows and of the uniqueness of crystalline flat flows is settled.

Giga & Pozar again

Shortly after, Giga & Pozar (to appear on CPAM) : crystalline viscosity approach in N-dimensions for

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In all cases covered by both methods, the two approaches yield the same solutions.

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- Unifying approach to deal with a general class of nonlocal curvature flows
- The variational point of view highlights the crucial role of convexity (submodularity)
- General consistency result between viscosity solutions and minimizing movements
- The general theory does not apply to the crystalline mean curvature flow
- New recent approach: provides the first general well-posedness result for crystalline mean curvature flows valid in any dimension and for arbitrary initial sets

Thank you for your attention!