Randomized final data problem for the nonlinear Schrödinger and the Gross-Pitaevskii equations

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(joint work with Takuto Yamamoto)

Equations

Goal: Asymptotic behavior as $t \to \infty$ of solutions for random data to the nonlinear Schrödinger equation:

$$(\textbf{NLS}) \quad i\dot{u} + \Delta u = -|u|^p u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{C} \quad (p > 0,\ d \in \mathbb{N}),$$

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for which the most basic class of solutions is $L^2(\mathbb{R}^d)$ in space:

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or slightly more concrete models, such as the nonlinear Schrödinger system

$$(\textbf{NSS}) \quad \begin{cases} i\dot{u}_1 + \Delta u_1 = \overline{u_2} u_3, \\ i\dot{u}_2 + \Delta u_2 = \overline{u_1} u_3, \\ i\dot{u}_3 + \Delta u_3 = u_1 u_2, \end{cases} \qquad u(t,x) : \mathbb{R}^{1+3} \to \mathbb{C}^3,$$

where u_1, u_2, u_3 describe the laser incident wave, the scattered wave, and the plasma electric field, respectively, in the Raman scattering.

Equations

Another case is also the (defocusing) NLS

$$\textbf{(GP)} \quad i\dot{u} + \Delta u = |u|^2 u, \quad u(t,x) : \mathbb{R}^{1+3} \to \mathbb{C},$$

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but with a non-zero background or boundary condition at $|x| \to \infty$, which is more suitable in superfluid, nonlinear optics, etc.
More precisely, we consider disturbance from a plane wave solution:

$$u(t,x) = ae^{i\omega t + i\xi x}(1 + v(t,x)), \quad a + \omega = -|\xi|^2,$$

with some parameters $a > 0 > \omega$ and $\xi \in \mathbb{R}^3$, and v decaying as $|x| \to \infty$.

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with some parameters $a > 0 > \omega$ and $\xi \in \mathbb{R}^3$, and v decaying as $|x| \to \infty$.
Using the invariance of equation, we can reduce it to the simplest case:

$$u(t,x) = e^{-it}(1 + v(t,x)).$$

We call the equation in this setting the Gross-Pitaevskii.
Note: $|u|^2 u$ contains $O(v^2)$, quadratic interactions on $\mathbb{R}^3$.

Scattering problem: nonlinearity v.s. dispersion

The (nonlinear) scattering theory aims at approximating a nonlinear solution u with interactions, e.g., for (**NLS**)

$$i\dot{u} + \Delta u = -|u|^p u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{C},$$

as $t \to \infty$ by a solution v without interaction to the 'free' equation:

$$i\dot{v} + \Delta v = 0, \quad v(t,x) : \mathbb{R}^{1+d} \to \mathbb{C},$$

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anticipating that the dispersion forces the wave to spread ($|x| \to \infty$), so nonlinear interactions ($|u|^p u$) have essential effect within finite time only. The most basic approximation is in $L^2(\mathbb{R}^d)$, namely

$$\|u(t) - v(t)\|_{L^2(\mathbb{R}^d)} \to 0 \quad (t \to \infty).$$

Then, writing $v(t) = e^{it\Delta}\varphi_+$ with the unitary group (i.e. $\varphi_+ \in L^2(\mathbb{R}^d)$ is the initial data for v), we call φ_+ the final data for u.

Scattering problem

The fundamental questions in the scattering problem are

1. Existence: $\forall\varphi_+$:final data, $\exists u$: nonlinear (global) solution?
2. Uniqueness: $\varphi_+ \mapsto u$?
3. Completeness: $\forall u$, $\exists\varphi_+$?

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In the case of (**NLS**) with φ_+ in $L^2(\mathbb{R}^d)$, we know

1. Existence: Yes, if $d \geq 3$ and $2/d < p < 4/d$. (N. '00)
2. Uniqueness: An open problem.
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For $p = 2/d$, the asymptotics need nonlinear modification (Ozawa '91).
For $4/d \leq p \leq 4/(d-2)$, blow-up occurs backward in time for some solutions asymptotically free near $t = \infty$ (cf. N.-Schlag '12).

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The scaling invariance of (**NLS**) shows that it is not the case for $p < 4/d$:

$$u(t,x) : (\mathbf{NLS}) \implies \forall \lambda > 0,\ u_\lambda(t,x) := \lambda^{2/p} u(\lambda^2 t, \lambda x) : (\mathbf{NLS}),$$

$$\|u_\lambda(0)\|_{L^2(\mathbb{R}^d)} = \lambda^{2/p-d/2} \|u(0)\|_{L^2(\mathbb{R}^d)} \to 0 \quad (\lambda \to +0).$$

By the scaling, the scattering problem under the restrictions

$$\|\varphi_+\|_{L^2} \le \varepsilon, \quad \|u(0)\|_{L^2} \le \varepsilon, \quad t \ge 1/\varepsilon$$

is equivalent (for any $\varepsilon > 0$) to that with

$$\|\varphi_+\|_{L^2} \le 1, \quad \|u(0)\|_{L^2} \le 1, \quad t \ge 1.$$

Hence neither the smallness of φ_+ and $u(0)$ nor largeness of t helps.

Supercriticality v.s. Randomization

Randomizing the data can break the supercriticality.
Burq-Tzvetkov ('08) considered the (rough) initial data problem for the nonlinear wave equation:

$$(\textbf{NLW}) \quad \ddot{u} - \Delta u = -u^3, \quad u(t,x) : \mathbb{R} \times M \to \mathbb{R},$$

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$$(u(0), \dot{u}(0)) = (f_1, f_2) \in H^s(M) \times H^{s-1}(M) : \text{Sobolev space}$$

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is a supercritical setting if $s < 1/2$. Randomizing the initial data

$$H^s \times H^{s-1} \ni (f_1, f_2) \mapsto (f_1^\omega, f_2^\omega) \in L^2(\Omega; H^s \times H^{s-1}),$$
$$(u(0), \dot{u}(0)) = (f_1^\omega, f_2^\omega),$$

in a probability space Ω, however, they proved unique existence of local solutions for almost every $\omega \in \Omega$ for $s \geq 1/4$.

Randomization for scattering

Recently, Murphy ('17 arxiv) employed the idea to tackle the supercritical scattering problem, namely $p < 4/d$ and $\varphi_+ \in L^2(\mathbb{R}^d)$ for (**NLS**)

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$\forall d \in \mathbb{N}$, $\exists p_0(d) \in (2/d, 4/d)$, $\forall p \in (p_0(d), 4/d)$, $\forall \varphi_+ \in L^2(\mathbb{R}^d)$, almost every $\omega \in \Omega$, $\exists! u$: sol. of (**NLS**) (in a certain function space), s.t.

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$$\|u(t) - e^{it\Delta}\varphi_+^\omega\|_{L^2(\mathbb{R}^d)} \to 0 \quad (t \to \infty).$$

Here $p_0(d)$ is called the Strauss exponent for (**NLS**):

$$p_0(d) = \frac{\sqrt{d^2 + 12d + 4} - d + 2}{2d},$$
$$p_0(1) = 2.56..., \quad p_0(2) = \sqrt{2}, \quad p_0(3) = 1, \quad p_0(4) = 0.78...$$

Quadratic nonlinearity on $\mathbb{R}^3$ is excluded.

How the randomization works?

Murphy defined the randomization $L^2(\mathbb{R}^d) \times \Omega \ni (\varphi, \omega) \mapsto \varphi^\omega$ as follows. Let $\{g_k(\omega)\}_{k\in\mathbb{Z}^d}$ be i.i.d. mean-0 Gaussian random variables, and $\chi \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$0 \le \chi, \quad \sum_{k\in\mathbb{Z}^d} \chi(x-k) = 1.$$

Then φ^ω is given by

$$\varphi^\omega(x) := \sum_{k\in\mathbb{Z}^d} g_k(\omega)\varphi_k(x), \quad \varphi_k(x) := \chi(x-k)\varphi(x).$$

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Note that each φ_k is compactly supported (localized) around $k \in \mathbb{Z}^d$.

Lührmann-Mendelson ('14) considered the above form of randomization in the Fourier transform, for the initial data problem of (**NLW**) on $\mathbb{R}^3$.

How the randomization works?

Each localized piece $\varphi_k \in L^1(\mathbb{R}^d)$ enjoys much better dispersion, e.g.,

$$\|e^{it\Delta}\varphi_k\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}\|\varphi_k\|_{L^1(\mathbb{R}^d)},$$

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than the original $\varphi \in L^2(\mathbb{R}^d)$. Randomization reduces interactions among them in average, through (Burq-Tzvetkov)

$$2 \le \forall\alpha < \infty,\ \forall c \in \ell^2(\mathbb{Z}^d), \quad \|\sum\nolimits_k g_k(\omega)c_k\|_{L^\alpha(\Omega)} \lesssim \sqrt{\alpha}\|c\|_{\ell^2(\mathbb{Z}^d)}.$$

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In particular, the Strichartz estimate gains more integrability (Murphy):

$$\|e^{it\Delta}\varphi^\omega\|_{L^\alpha_\omega L^q_t L^r_x(\Omega\times(1,\infty)\times\mathbb{R}^d)} \lesssim \sqrt{\alpha}\|\varphi\|_{L^2(\mathbb{R}^d)},$$
$$\frac{1}{q} > \frac{d}{2} - \frac{d}{r}, \quad 2 \le q, r \le \alpha < \infty.$$

The deterministic case (w/o ω) is only for $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$, $2 \le q, r \le \infty$.

Main result

We extend Murphy's result to lower powers p.

Theorem (N.-Yamamoto '18)

$\forall d \in \mathbb{N}$, $\exists p_1(d) \in (2/d, p_0(d))$, $\forall p \in (p_1(d), 4/d)$, $\forall \varphi \in L^2(\mathbb{R}^d)$, *almost every* $\omega \in \Omega$, $\exists! u$*: sol. of* (**NLS**) *(in another function space), s.t.* $\|u(t) - e^{it\Delta}\varphi_+^\omega\|_{L^2(\mathbb{R}^d)} \to 0$ *as* $t \to \infty$.

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In particular, since $p_1(3) < 1$, we can treat quadratic interactions in $\mathbb{R}^3$, such as (**NSS**), as well as (**GP**).

$$p_1(d) = \frac{\sqrt{d^2+24d+16} - d + 4}{4d}, \quad p_0(d) = \frac{\sqrt{d^2+12d+4} - d + 2}{2d},$$

$$p_1(1) = 2.35..., \quad p_1(2) = 1.28..., \quad p_1(3) = 0.90..., \quad p_1(4) = 0.70...,$$
$$p_0(1) = 2.56..., \quad p_0(2) = 1.41..., \quad p_0(3) = 1, \quad p_0(4) = 0.78...$$

Critical exponents: Scaling and Dispersion

(**NLS**) is invariant for the scaling

$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{2/p} u(\lambda^2 t, \lambda x) \quad (\lambda > 0),$$
$$\|u_\lambda(0)\|_{L^q_x(\mathbb{R}^d)} = \|u(0)\|_{L^q_x(\mathbb{R}^d)} \iff q = dp/2.$$

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Dispersive estimate for the free Schrödinger evolution is

$$v(t) = e^{it\Delta}\varphi, \ 1 \le r \le 2 \implies \|v(t)\|_{L^{r^*}_x(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}+\frac{d}{r}} \|\varphi\|_{L^r_x(\mathbb{R}^d)}.$$

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The four critical exponents are characterized by

1. Fujita exponent: $p = 2/d \iff q = 1$.
2. mass-critical exponent: $p = 4/d \iff q = 2$.
3. Strauss exponent: $p = p_0(d) \iff q = (p+2)^*$. For $t > 0$,
$$\varphi \in L_x^{(p+2)^*} \implies v(t) \in L_x^{p+2} \implies |v(t)|^p v(t) \in L_x^{(p+2)^*}.$$
4. Our exponent: $p = p_1(d) \iff q = (2p+2)^*$. For $t > 0$,
$$\varphi \in L_x^{(2p+2)^*} \implies v(t) \in L_x^{2p+2} \implies |v(t)|^p v(t) \in L_x^2.$$

The result for (GP): Randomized in the energy space

Around the plane waves, the (renormalized) L^2 is no longer positive, but

$$u = e^{-it}(1+v), \quad E(u) := \int_{\mathbb{R}^3} \frac{|\nabla u|^2}{2} + \frac{(|u|^2-1)^2}{4} dx > 0.$$

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We need a $\mathbb{C}$-linearization to get a 'free' unitary group:

$$w := \sqrt{2-\Delta}\,\mathrm{Re}\, v + i\sqrt{-\Delta}\,\mathrm{Im}\, v, \quad E(u) \approx \|w\|^2_{L^2(\mathbb{R}^2)}.$$

Then, $\forall \varphi \in L^2(\mathbb{R}^3)$, a.s.$\omega \in \Omega$, $\exists! u$: sol. of (**GP**) s.t.,

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In the deterministic case, the existence is by Gustafson-N.-Tsai ('09), but we need in general a quadratic correction term in the asymptotic formula. For φ_+^ω we do not need it, because $\varphi_+^\omega \in \dot{H}^s(\mathbb{R}^d)$ a.s., for $s > -3/2$.

Open problem: Global dynamics from Random initail data

Consider the randomized initial data problem for (**NLS**)

$$i\dot{u} + \Delta u = -|u|^p u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{C} \quad (d \in \mathbb{N},\ 2/d < p < 4/d),$$
$$u(0,x) = \varphi^\omega(x), \quad (\varphi \in L^2(\mathbb{R}^d), \quad \omega \in \Omega).$$

Since $\varphi^\omega \in L^2(\mathbb{R}^d)$ a.s., we have a global solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^d))$ (Tsutsumi '87). What is its asymptotic behavior as $t \to \infty$?

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Randomized soliton resolution conjecture

For almost every $\omega \in \Omega$, there is a sequence of solitons (maybe infinite) $u_j(t,x) = e^{it\omega_j + ix\xi_j}\varphi_j(x - c_j t)$ and $\varphi_+ \in L^2(\mathbb{R}^d)$ such that

$$\|u(t) - \sum_j u_j(t) - e^{it\Delta}\varphi_+\|_{L^2(\mathbb{R}^d)} \to 0 \quad (t \to \infty).$$