Scattering for the 3D Gross-Pitaevskii equation

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Outline

1. Introduction and results

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2. **Proof of the theorem**
   - **Difficulty 1: singularity at zero frequency**
   - **Difficulty 2: 3D quadratic term**
   - **Difficulty 3: weak low-frequency component of u_2**

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Gross-Pitaevskii (GP) equation

Consider the Gross-Pitaevskii (GP) equation

$$i\psi_t + \Delta\psi = (|\psi|^2 - 1)\psi, \quad \psi : \mathbb{R}^{1+3} \to \mathbb{C} \tag{1}$$

with the boundary condition

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The non-vanishing boundary condition: **very different**.
Physical contexts: Bose-Einstein condensates, superfluids and nonlinear optics, or in the hydrodynamic interpretation of NLS

Energy conservation

Let $u = \psi - 1$. Then u satisfies zero boundardy condition and

$$i\partial_t u + \Delta u - 2\,\mathrm{Re}\, u = u^2 + 2|u|^2 + |u|^2 u \tag{4}$$

which is equivalent to (writing $u = u_1 + iu_2$)

$$\begin{aligned} \dot{u}_1 &= -\Delta u_2 + 2(u_1 + |u|^2/2)u_2, \\ -\dot{u}_2 &= (2 - \Delta)u_1 + 3u_1^2 + u_2^2 + |u|^2 u_1. \end{aligned} \tag{5}$$

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Conservation of the energy:

$$\begin{aligned} E(u) &:= \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{(|u|^2 + 2\,\mathrm{Re}\,u)^2}{2} dx \\ &= \int_{\mathbb{R}^3} |\nabla \psi|^2 + \frac{(|\psi|^2 - 1)^2}{2} dx = E(u_0). \end{aligned} \tag{6}$$

Note. We do not have conservation of $\|u\|_2^2$, but $\|u(t)\|_2 \le Ce^{Ct}$ by Gronwall inequality.

Well-posedness

- In Zhidkov space $X^k = \{u \in L^\infty : \partial^\alpha \in L^2, 1 \le |\alpha| \le k\}$
  $d = 1$: Zhidkov 1987;
  $d = 2, 3$: Gallo 2004
- In H^1, GWP
  $d = 2, 3$: Béthuel and Saut 1999
- In energy space, GWP
  $d = 1, 2, 3$ and small data for $d = 4$: Gérard;
  $d = 4$: Killip-Oh-Pocovnicu-Visan

Remark. No scattering in the above works.

GWP in energy space

Energy space

$$\mathbb{E} := \{f \in \dot{H}^1(\mathbb{R}^3) : 2\,\mathrm{Re}\, f + |f|^2 \in L^2(\mathbb{R}^3)\} \tag{7}$$

with the distance $d_{\mathbb{E}}(f, g)$ defined by

$$d_{\mathbb{E}}(f, g)^2 = \|\nabla(f - g)\|_{L^2}^2 + \frac{1}{2}\big\||f|^2 + 2\,\mathrm{Re}\, f - |g|^2 - 2\,\mathrm{Re}\, g\big\|_{L^2}^2.$$

Note that $(\mathbb{E}, d_{\mathbb{E}})$ is a complete metric space.

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Theorem (Gérard, 2006)

Unconditional GWP of (4) in the energy space $\mathbb{E}$.

Remark. GWP in $H^1 \subset \mathbb{E}$, Béthuel-Saut, 1999.
method: LWP by Strichartz analysis + energy conservation

Traveling wave solutions

There are a family of the solutions $\psi(t,x) = v_c(x-ct)$ with finite energy for $0 < |c| < \sqrt{2}$.
Moreover, Béthuel-Gravejat-Saut 2009 proved that in 3D

$$E^* := \inf\{E(\psi - 1) | 1 \neq \psi(t,x) = v(x-ct) \text{ solves (1) for some } c\} > 0,$$

conjectured that E^* is the threshold for the global dispersive solutions.

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Remark.
1. In 2D, there is no lower bound.
2. In the radial case, are there special non-scattering solutions in energy space? Not clear.

Scattering

In 3D, scattering was proved by Gustafson-Nakanishi-Tsai (2006,2007,2009) for small data in weighted Sobolev space.

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Theorem (G.-Hani-Nakanishi, 2017)

For 3D GP equation, scattering holds for small radial data in $\mathbb{E}$.

Remark. Radial symmetry can be replaced by additional one order angular regularity.

Remaining questions: How about non-radial case? What about large data? Is smallness not needed?

Let's state our theorem more precisely.

By the diagonalising transform

$$u = u_1 + iu_2 \longrightarrow v = v_1 + iv_2 := u_1 + iUu_2, \tag{8}$$

with

$$U := \sqrt{-\Delta(2-\Delta)^{-1}},$$

we can rewrite the GP equation for v:

$$i\partial_t v - Hv = U(3u_1^2 + u_2^2 + |u|^2 u_1) + i(2u_1 u_2 + |u|^2 u_2), \tag{9}$$

where (one can write $u_1 = v_1,\ u_2 = U^{-1}v_2$)

$$H := \sqrt{-\Delta(2-\Delta)}. \tag{10}$$

Note. H has symbol $|\xi|\sqrt{2+|\xi|^2}$. It behaves as Schrödinger equation for high frequency and as wave equation for low frequency.

Theorem

There exists $\delta > 0$ such that for any $u_0 \in \mathbb{E}$, radial, with $E(u_0) \leq \delta$, there exists a unique global solution $u \in C(\mathbb{R} : \mathbb{E})$ to (4). Moreover, there exists $\phi_\pm \in H^1$ such that

$$\lim_{t \to \pm\infty} \left\| u_1 + (2 - \Delta)^{-1} u_2^2 + iUu_2 - e^{-itH} \phi_\pm \right\|_{H^1} = 0. \qquad (11)$$

Remark.

We have the decay for quadratic terms of u_1 (not true for u_2):

$$\lim_{t \to \pm\infty} \|(2 - \Delta)^{-1} u_1^2\|_{H^1} = 0.$$

We can transfer the asymptotic behaviour in the energy space $\mathbb{E}$. Indeed, we can prove

$$\lim_{t \to \pm\infty} d_{\mathbb{E}}(u, T^{-1}(e^{-itH} \phi_\pm)) = 0,$$

where $T(u) = T(u_1 + iu_2) = u_1 + (2 - \Delta)^{-1} u_2^2 + iUu_2$.

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Difficulty 1: singularity at zero frequency

At low frequency $U^{-1} \approx |\nabla|^{-1}$.
Idea. Nonlinear transform

This was treated by Gustafson-Nakanishi-Tsai using some nonlinear transform

$$z = z_1 + iz_2 = u_1 + \frac{u_1^2 + u_2^2}{2 - \Delta} + iUu_2 \tag{12}$$

Under the transform (12)

$$\begin{aligned} iz_t - Hz = & -2iU(u_1^2) - 4\langle\nabla\rangle^{-2}\nabla \cdot (u_1 \nabla u_2) \\ & + [-iU(|u|^2 u_1) + U^2(|u|^2 u_2)]. \end{aligned} \tag{13}$$

Note. Quadratic terms have no zero frequency singularity.

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Scattering results for $p < \frac{4}{3}$:

- $0 < p \leq \frac{2}{3}$
  Any nontrivial solution u, with $\phi \in \mathcal{S}$, does not scatter to linear solution in L^2. (Glassey 1974, Strauss 1974)
- $\frac{2}{3} < p < \frac{4}{3}$
  For any $\phi \in H^1$ with $\|x\phi\|_2 < \infty$, the solution u scatters to linear solution in L^2. (Tsutsumi-Yajima, 1984)
  Nakanishi-Ozawa 2002, Masaki 2015
- $p = \frac{2}{3}$
  Modified scatterings occur. Ozawa, Hayashi, Naumkin, etc

Note. For quadratic terms $2/3 < 1 < 4/3$
Weighted H^s is usually needed for 3D quadratic terms.

Our ideas. Replace weighted H^s by additional angular regularity. In particular, in the radial case, we can handle H^s.

Use earlier ideas in the works on 3D Zakharov system
(G.-Nakanishi 2012, G.-Lee-Nakanishi-Wang 2015)

To illustrate our ideas, we consider the classical 3D quadratic Klein-Gordon equation

$$u_{tt} - \Delta u + u = u^2 \tag{14}$$

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- Klainerman's vector field method (1985)
  Existence of enough vector fields + energy estimates
- Shatah's normal form method (1985)
  Non-resonant structure + normal form transform (transfer the quadratic term to a cubic or higher order term).
  Key non-resonance: $\langle \xi \rangle + \langle \eta \rangle - \langle \xi + \eta \rangle \gtrsim \frac{1}{\langle \eta \rangle}$

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Our approach: generalized Strichartz estimates+(partial) normal form

$$(i\partial_t + \langle D\rangle)u = \langle D\rangle^{-1}u^2$$

Let $K(t) = e^{it\langle D\rangle}$. Then

Lemma 1 (radial case) (G.-Nakanishi-Wang 2013).

Let $r > 10/3$, for ϕ radial

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2L_x^r}\lesssim 2^{k\beta_k}\|\phi\|_{L_x^2},$$

where $P_k \approx \mathcal{F}^{-1}1_{|\xi|\sim 2^k}\mathcal{F}$ and

$$\beta_k = \begin{cases} \frac{1}{2} - \frac{3}{r}, & k < 0; \\ 1 - \frac{3}{r}, & k \geq 0; \frac{10}{3} < r < 4; \\ \frac{1}{4} + \epsilon, & k \geq 0; r = 4; \\ \frac{1}{r}, & k \geq 0; r > 4. \end{cases}$$

Remark. The classical best non-radial estimates

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2L_x^6}\lesssim 2^{5k/6}\|\phi\|_{L_x^2}.$$

Wave: Klainerman-Machedon, Sogge, Sterbenz 2005

Schrödinger: Sogge, Shao, G.-Wang 2010

Lemma 2 (non-radial case) (G.-Hani-Nakanishi 2016).
Let $r > 10/3$,

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2L_x^rL_\sigma^2} \lesssim B_k(2,r)\|\phi\|_{L_x^2},$$

where $P_k \approx \mathcal{F}^{-1}1_{|\xi|\sim 2^k}\mathcal{F}$ and

$$B_k(2,r) = \begin{cases} 2^{k(\frac{1}{2}-\frac{3}{r})}, & k < 0; \\ 2^{k(1-\frac{3}{r})}, & k \geq 0; \frac{10}{3} < r < 4; \\ \langle k\rangle 2^{\frac{k}{4}}, & k \geq 0; r = 4; \\ 2^{\frac{k}{r}}, & k \geq 0; r > 4. \end{cases}$$

Remark.
Wave: Sterbenz 2005
Schrödinger: G.-Lee-Nakanishi-Wang 2014, G. 2016
Theorem. Scattering for 3D quadratic KG with small data in $H^1 \times L^2$ and with 1-order angular regularity.

Generalized Strichartz for GP

Lemma 3 (G.-Hani-Nakanishi 2016). For GP, we have: for $r > 10/3$,

$$\|e^{-itH}P_k\phi\|_{L_t^2 L_x^r L_\sigma^2} \lesssim C_k(2,r)\|\phi\|_{L_x^2(\mathbb{R}^3)}, \tag{15}$$

where

$$C_k(2,r) = \begin{cases} 2^{k(\frac{1}{2}-\frac{3}{r})}, & k \geq 0; \\ 2^{k(2-\frac{7}{r})}, & k < 0, \frac{10}{3} < r < 4; \\ 2^{k(1-\frac{3}{r})}, & k < 0, r > 4; \\ \langle k \rangle 2^{\frac{k}{4}}, & k < 0, r = 4. \end{cases} \tag{16}$$

Note. $H := \sqrt{-\Delta(2-\Delta)}$.

Intermediate Theorem

With this estimate and the nonlinear transform by (12), we get

Theorem

Scattering holds for small radial data $u(0) \in H^1$.

Question: What about energy space $\mathbb{E}$? For small data, we can think

$$\mathbb{E} = \{u : \mathrm{Re}\, u \in H^1, \mathrm{Im}\, u \in \dot{H}^1 \cap L^4\}$$

Difficulty. $\mathrm{Im}\, u \notin L^2$.

Outline

Difficulty. For $u(0) \in \mathbb{E}$, we can only have $\nabla u_2(0) \in L^2$.
Recall under the nonlinear transform

$$z = z_1 + iz_2 = u_1 + \frac{u_1^2 + u_2^2}{2 - \Delta} + iUu_2 \tag{17}$$

$$\begin{aligned} iz_t - Hz = & -2iU(u_1^2) - 4\langle\nabla\rangle^{-2}\nabla \cdot (u_1 \nabla u_2) \\ & + [-iU(|u|^2 u_1) + U^2(|u|^2 u_2)]. \end{aligned} \tag{18}$$

Problematic terms: $u_2^2 \Delta u_2$ or $u_2^2 u_1$ when u_2 has very low frequency.

Key new ingredients: "Null-structure" achieved by new nonlinear transform. Let

$$m = m_1 + im_2 = u_1 + \frac{2u_1^2 + u_2^2}{2 - \Delta} + iUu_2 \tag{19}$$

Then

$$i\partial_t m - Hm = N_2(m) + N_3(m, u) + N_4(m, u) + N_5(m, u)$$

where

$$N_2(m,u) = U(m_1^2) + \frac{2i}{2-\Delta}[-3m_1\Delta u_2 - 2\nabla m_1 \cdot \nabla u_2],$$

$$N_3(m,u) = U(2m_1 R) + iN_3^1(u) + \frac{2i}{2-\Delta}[4u_1 m_1 u_2 - m_1^2 u_2],$$

$$N_4(m,u) = U(R^2 - |u|^4/4) + \frac{2i}{2-\Delta}[4u_1 R u_2 - 2u_2 m_1 R],$$

$$N_5(m,u) = \frac{2i}{2-\Delta}[-u_2 R^2 + u_2|u|^4/4],$$

with

$$R = \frac{-\Delta u_2^2}{2(2-\Delta)} - \frac{(2+\Delta)u_1^2}{2(2-\Delta)},$$

$$N_3^1(u) = (2-\Delta)^{-1}\left\{-2u_2|\nabla u_2|^2 + \frac{3\Delta u_2^2}{2-\Delta}\Delta u_2 + \frac{2\nabla\Delta u_2^2}{2-\Delta}\nabla u_2\right\} + \frac{4}{2-\Delta}\left[\frac{2u_1^2}{2-\Delta}\Delta u_2\right] - \left[\frac{2+\Delta}{2-\Delta}u_1^2\right]u_2.$$

Remark

Our nonlinear transform can achieve cancellation and is actually natural.

The GP equation (4) for $u = u_1 + iu_2$ can be rewritten as follows

$$\begin{aligned} \dot{u}_1 &= -\Delta u_2 + 2(u_1 + |u|^2/2)u_2, \\ -\dot{u}_2 &= (2-\Delta)u_1 + 3u_1^2 + u_2^2 + |u|^2 u_1 \\ &= (2-\Delta)(u_1) + 2u_1^2 + u_2^2 + (2u_1 + |u|^2)^2/4 - |u|^4/4. \end{aligned}$$

Note that $2u_1 + |u|^2 \in L^2$ is bounded by the conserved energy.
In view of the equation of u_2, we first make the following change of variables

$$z_1 := u_1 + \frac{2u_1^2 + u_2^2}{2-\Delta}, \quad z_2 = u_2.$$

Thank you very much!